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On behaviour of solutions for a nonlinear viscoelastic equation with variable-exponent nonlinearities

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ABSTRACT

This paper deals with a viscoelastic wave equation with variable-exponent nonlinearities, subject to nonlinear boundary feedback. Under appropriate conditions, a general decay result associated to solution energy is proved. It is also shown that regarding arbitrary positive initial energy, solutions blow-up in a finite time.

1. Introduction

This article investigates the solution behaviour of the following viscoelastic problem:

$$u_{tt} - \Delta u - div(|\nabla u|^{m(x)}\nabla u) + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + h(x,t,u,\nabla u) + \beta u_t$$
$$= |u|^{p(x)}u, \quad in \;\; \Omega \times (0,+\infty) \tag{1}$$

$$\begin{cases} u(x,t) = 0, & x \in \Gamma_0, t > 0 \\ \dfrac{\partial u}{\partial n}(x,t) = \displaystyle\int_0^t g(t-\tau)\frac{\partial u}{\partial n}(\tau)d\tau - |\nabla u|^{m(x)}\frac{\partial u}{\partial n} + \alpha u, & x \in \Gamma_1, t > 0 \end{cases} \tag{2}$$

$$u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), \quad in \;\; \Omega, \tag{3}$$

where Ω is a bounded domain in $R^N (N \geq 1)$ with a smooth boundary $\Gamma_0 \cup \Gamma_1 = \partial\Omega$. Here, $\alpha, \beta \geq 0$ are constants and the exponents $m(.)$ and $p(.)$ are given measurable functions on $\overline{\Omega}$ such that:
(A1)

$$0 \leq m^- \leq m(x) \leq m^+$$

$$0 \leq p^- \leq p(x) \leq p^+$$

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with

$$m^- := essinf_{x\in\overline{\Omega}} m(x), \quad m^+ := esssup_{x\in\overline{\Omega}} m(x),$$

$$p^- := essinf_{x\in\overline{\Omega}} p(x), \quad p^+ := esssup_{x\in\overline{\Omega}} p(x).$$

(A2) The kernel of memory, $g : R^+ \to R^+$, is a differentiable and non-increasing function satisfying

$$g(0) > 0, \quad 1 - \int_0^{+\infty} e^{-\lambda s} g(s)ds = l > 0,$$

for some positive λ.
(A3) $h(x, t, u, \nabla u)$ is a function that satisfies

$$|h(x, t, u, \nabla u)| \le M_1|u| + M_2|\nabla u|,$$

for some positive M_1, M_2.

It is known that viscoelastic materials show natural damping properties, which is due to the special property of these substances in keeping memory of their past history. But modelling of some physical phenomena such as flows of electro-rheological fluids, nonlinear viscoelasticity and image processing give rise to equation with nonstandard growth conditions, that is, equations with variable exponents of nonlinearities. These models include hyperbolic, parabolic or elliptic equations that are nonlinear in gradient of the unknown solution and with variable exponents of nonlinearity. More details on these problems can be found in [1–5]

Before going any further, it is worth pointing out some classical results. In bounded domains, there is an extensive literature on the existence, asymptotic behaviour and nonexistence of solution for the following equation

$$u_{tt} - \Delta u + h(u_t) = f(u), \quad in \ \ \Omega \times (0, +\infty)$$

In the absence of the source term i.e. ($f = 0$), it is well known that the damping term $h(u_t)$ assures global existence and decay of the solution energy for arbitrary initial data (see [6,7]). In contrast, in the absence of the damping term, the source term causes finite-time blow-up of solutions with a large initial data (negative initial energy) (see [8]). The interaction between the damping term and the source term was first considered by Levine [9] in the linear damping case $h(u_t) = au_t$ and a polynomial source term of the form $f(u) = b|u|^{(p-2)}u$. He showed that solutions with negative initial energy blow up in finite time (see also [10]). Georgiev and Todorova [11] extended Levine's result to the nonlinear damping case $h(u_t) = a|u_t|^{(m-2)}u_t$ with Dirichlet boundary conditions. They showed that solutions with negative energy continue to exist globally in time if $m \ge p \ge 2$ and blow up in finite time if $p \ge m \ge 2$ and the initial energy is sufficiently negative. Bilgin and Kalantarov [12] investigated blow up of solutions for the following $m-$Laplacian initial–boundary value problem:

$$u_{tt} - \nabla[(a_0 + a|\nabla u|^{m-2})\nabla u] - b\Delta u_t = g(x, t, u, \nabla u) + |u|^{p-2}u, \quad x \in \Omega, \ t > 0,$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \ t > 0,$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega.$$

They obtained sufficient conditions on initial functions for which there exists a finite time that some solutions blow up at this time. In all of these papers, the authors obtained the global existence and the finite time blow up of solutions by potential well theory combined with Galerkin method and concavity method.

In the presence of memory term, Fabrizio and Polidoro [13] studied the following equation with homogeneous boundary condition

$$u_{tt} - \Delta u + \int_0^t \Delta u(\tau)d\tau + u_t = 0 \quad in \ \Omega \times (0, \infty)$$

and showed that the exponential decay of the relaxation function is a necessary condition for the exponential decay of the solution energy. Cavalcanti and Oquendo [14] considered the following equation

$$u_{tt} - k_0\Delta u + \int_0^t div[a(x)g(t-\tau)\nabla(x, \tau)]d\tau + b(x)h(u_t) + f(u) = 0$$

for $a(x) + b(x) \ge \rho > 0$. An exponential stability result has been established when g decays exponentially and h is a linear function. Moreover, a polynomial stability result for g decaying polynomially and nonlinear h has been proved.

Messaoudi [15] considered the following initial–boundary value problem:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m-2}u_t = |u|^{p-2}u, \quad in \ \ \Omega \times (0, +\infty)$$

$$u(x, t) = 0 \ \ on \ \ \partial\Omega$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad in \ \ \Omega$$