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Theoretical aspects of a binary mixture flow

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ABSTRACT

We study existence and long-time asymptotic behaviour of non-negative weak solutions for the coupled system of nonlinear partial differential equations. The system models dynamics of a binary mixture flow in the lubrication approximation regime. The applications of results include the process of drying of multi-component paint and distribution of swarming bacteria population. We also present analytical estimations of the dry out time.

1. Introduction

Coating of substrates with thin layers of paint is a general industrial process. A mathematical model based on classical lubrication theory was derived in [1] to study variations in layer thickness which occur as a paint layer dries. The effects of variable surface tension, viscosity, solvent diffusivity and solvent evaporation rate were considered. The authors analysed the time evolution of small perturbations to the thickness of a drying paint layer and also to the concentration of solvent, and provided an analytical description of the reversal effect of an initial perturbation to the thickness of the layer.

Three-dimensional numerical simulations based on the lubrication approximation for the flow of drying paint films on horizontal substrates were presented in [2]. The paint was modelled as a multi-component liquid with one non-volatile and one volatile component which the authors called the resin and the solvent respectively and a closed-form linearised solution was found for coating layers that are of almost uniform thickness. Numerical simulations were used to analyse the effect of surface tension gradients due to compositional changes in a three-dimensional flow field and the authors suggested new methods of using these gradients to obtain a nearly uniform final coating layer. At approximately the same time the authors of [3] modelled paint coating on different substrate shapes and applied not only change of wetting properties but also viscosity dependence on flow stress and mixture composition.

In [4] a mathematical model of the cratering of paint during drying was derived in the presence of a surfactant. Nonuniform surfactant on the free surface of the thin film paint created surface tension gradients, and the resulting shear stresses generated the outward flow which was responsible for crater formation. Because paint films were considered to be thin, lubrication theory was used to reduce the governing equations to three coupled partial differential equations. The additional effects of topography on the free surface were taken into account in [5] and the resulting time-dependent, nonlinear, coupled set of governing equations was solved using a full approximation storage multigrid method.

A two-phase model for a fluid composed of a volatile solvent and a non-volatile polymer was derived and studied in [6]. The model accounted for density differences between the phases as well as evaporation at a fluid–air interface. In the

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one dimensional case the authors explored the connection between evaporation and compositional buoyancy; the former promoted the growth of a polymer-rich skin at the free surface while the latter tended to pull the denser polymeric phase to the substrate.

Rupture is a non-linear instability resulting in a finite-time singularity as a film layer approaches zero thickness at a point. The dynamics of rupture in a generalised mathematical model of thin films of viscous fluids with modified evaporative effects was studied in [7]. A bifurcation diagram for rupture regimes was obtained from the model. The finite-time singularities were studied using asymptotic analysis accompanied by adaptive high resolution partial differential equation simulations.

Certain binary liquid mixtures have surface tension values that vary significantly with the ratio of the components [8]. A general example is an alkyd paint whose surface tension increases as the solvent evaporates. Strong surface tension gradient effects can arise for a thin, non-uniform coating layer of the mixture. Often such paint exhibits a surprising behaviour; an initial hump in the coating may turn into a local depression in the final dry coating [2,8]. An extended thin film model can capture such phenomena. Representative equations for this model (see [2]) are:

$$h_t = -\mathrm{div}\mathbf{Q} - E, \tag{1.1}$$

$$(c\,h)_t = \mathrm{div}(Dh\nabla c - \mathbf{Q}c), \tag{1.2}$$

in $Q_T = \Omega \times (0,T)$ coupled with boundary conditions

$$\nabla c \cdot \mathbf{n} = \nabla h \cdot \mathbf{n} = \nabla \Delta h \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N = 1, 2$, and initial data (h_0, c_0) are non-negative,

$$\mathbf{Q} = \frac{h^2}{2\mu}\nabla\sigma - \frac{h^3}{3\mu}\nabla p, \;\; p = -\sigma\Delta h + \varrho g h,$$
$$E = E_0 f\,(1-c)_+^n, \;\; n \in [0,1], \;\; \sigma = \sigma_s + (\sigma_r - \sigma_s)c.$$

Here $h = h(x,y,t)$ is the coating height; $c = c(x,y,t)$ is the resin concentration of the bulk liquid; $h_r = h\,c$ is an imaginary thickness of the resin layer, which would be the coating thickness at a given station if all the solvent in the mixture evaporated; ϱ is the density; g is the acceleration of gravity; $\mu = \mu(c)$ is the depth-averaged viscosity; $D = D(c)$ is the diffusivity; E is the evaporation rate; σ_s and σ_r are the surface tension of pure solvent and pure resin, respectively; $f = f(x,y,t,h)$ is a dimensionless function that takes into account spatial and temporal variations in the evaporation rate not due to resin concentration; E_0 is a constant having units of velocity; n is an empirical exponent to model the diminishing behaviour of the evaporation rate as the resin fraction increases.

Similar thin-film equations also appear in mathematical biology contexts when dynamics of swarming colonies of bacteria is studied. For example the rate of expansion of bacterial colonies was investigated in terms of a mathematical model that combines biological as well as hydrodynamic processes in [9]. The model incorporated aspects of thin film flow with variable suspension viscosity, wetting, and cell differentiation. In [10] the authors developed a mathematical model for osmotically driven extraction of water and a Marangoni-driven expansion of the liquid film at the edge of the bacterial colony. Many swarming bacteria are aided by the production of a surfactant that lowers surface tension of the liquid film to improve bacterial motility. Because the depth of a typical colony is much smaller than its extent the authors used the following thin viscous liquid film equation to describe the liquid layer (see [10]):

$$h_t = -\nabla \cdot \Big(\frac{\sigma_m}{3\mu}\, h^3 \nabla\nabla^2 h + \frac{h^2}{2\mu}\,\nabla\sigma\Big) + E\,h.$$

Here, h is the thickness of the liquid film, the surface tension σ (σ_m is the limiting threshold value) depends on the concentration Γ of extracellular lipid rhamnolipid that is produced by bacteria and acts as a soluble surfactant agent to reduce surface tension in bacterial suspensions. E represents the rate of extracting water from the substrate by osmotic effects of rhamnolipid and is proportional to its local concentration.

The paper is structured as follows: in the second section we obtain spatially homogeneous solutions for the problem (1.1)–(1.3); in the third section we give weak formulation for the problem and we state our main result — the existence of weak solutions; in the fourth section we derive auxiliary estimates and finally prove the existence and some properties of the solutions. We also analyse long time asymptotic behaviour for some particular range of parameters and initial data.

2. Spatially homogeneous solutions

Multiple numerical simulations (see for example [2]) suggest that solutions of the problem (1.1)–(1.3), after some time, tend to behave as spatially homogeneous. In this section we examine existence of such type of the solutions. Later in Section 3, for a general case we prove convergence to spatially homogeneous solutions.

We assume that in (1.1), (1.2):

$$\sigma = \sigma_0 > 0, \;\; \mu = \mu_0 > 0, \;\; f(x,y,t,h) = h^m, \;\; m \geqslant 0,$$
$$D(c) = D_0(1-c)_+^l, \;\; l \geqslant 0.$$