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A bounded and efficient scheme for multidimensional problems with anomalous convection and diffusion

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ABSTRACT

This work is motivated by a generalization of the well-known Burgers–Fisher and Burgers–Huxley equations in multiple dimensions, considering Riesz fractional diffusion and convection. Initial–boundary conditions which are positive and bounded are imposed on a closed and bounded rectangular domain. In this manuscript we propose a finite-difference method to approximate the positive and bounded solutions of the fractional model. The methodology is a linear three-steps Crank–Nicolson technique which is based on the use of fractional centered differences. The properties of fractional centered differences are employed to establish the existence and the uniqueness of solutions of the finite-difference method, as well as the capability of the technique to preserve the positivity and the boundedness of the approximations. We show in this work that the method is capable of preserving some of the constant solutions of the continuous model. Additionally, we prove that our technique is a second-order consistent, stable and quadratically convergent scheme. Suitable bounds for the numerical solutions are also derived in this work. Finally, some illustrative simulations show that the method is able to preserve the positivity and the boundedness of the numerical approximations, in agreement with the analytic results proved in this work. Numerical comparisons provided in this work confirm the rate of convergence of the numerical technique.

1. Introduction

In recent years, the investigation of systems consisting of particles with long-range interactions has witnessed a notorious development in view of the various applications to the physical sciences. Indeed, beyond the classical examples of point masses in gravitational fields or systems of charged particles in space [1], there are various nontrivial physical systems of particles with long-range interactions. For instance, the nonlinear interactions of vortexes in two-dimensions, the elasticity arising from the study of planar stress, systems that consider dipolar forces [2] and the activation/repression of transcription in chromosomal and gene regulation [3] are some well known problems involving globally interacting particles. These and other examples have motivated the physical, the mathematical and the numerical investigation of this type of systems. Moreover, it is worth pointing out that the mathematical investigation of some models with long-range interactions has been extended to the continuous scenario (based on a continuous-limit process involving the Fourier series transform, the limit when the inter-particle distance tends to zero and the inverse Fourier transform), obtaining thus models with Riesz fractional derivatives in space [4].

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To be more precise, let $\alpha > 0$ and consider a physical system of interacting particles whose dynamics is described by the equations of motion

$$\frac{du_n}{dt}(t) = I_n(u(t)) + F(u_n(t)), \quad \forall t \in \mathbb{R}^+, \forall n \in \mathbb{Z}. \tag{1.1}$$

The functions u_n represent displacements from the equilibrium, F represents the interaction of the oscillators with an external force, and the distance between consecutive oscillators is equal to h. In general, we let I_n be given by

$$I_n(u(t)) = \sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} J(n, m)\left[u_n(t) - u_m(t)\right], \quad \forall t \in \mathbb{R}^+, \forall n \in \mathbb{Z}, \tag{1.2}$$

and $J \in L^2(\mathbb{Z})$ satisfies $J(n, m) = J(n - m) = J(m - n)$ for all $m, n \in \mathbb{Z}$. Let

$$J_\alpha(k) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{-ikn} J(n), \quad \forall k \in \mathbb{R}, \tag{1.3}$$

satisfy

$$A_\alpha = \lim_{k\to 0} \frac{J_\alpha(k) - J_\alpha(0)}{|k|^\alpha} \in \mathbb{R} \setminus \{0\}. \tag{1.4}$$

Taking Fourier series transform on both sides of (1.1), letting $h \to 0$ and obtaining inverse Fourier transform yields

$$\frac{\partial u}{\partial t}(x, t) - h^\alpha A_\alpha \frac{\partial^\alpha u}{\partial |x|^\alpha}(x, t) - F(u(x, t)) = 0, \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}^+. \tag{1.5}$$

Here, the fractional derivative in space of order α is understood in the sense of Riesz [4].

In summary, the use of Riesz fractional derivatives in the modeling of physical problems through partial differential equations is justified mathematically in the continuous limit of certain particle systems. Moreover, various fractional models from science and engineering are also capable of preserving some physical quantities. As examples, we may consider some gradient and Hamiltonian extensions of the Helmholtz conditions for phase space and some fractional equivalents of the Fokker–Planck equation for fractal media [5], continuous-limit approximations of systems of coupled oscillators with power-law interactions [6] and mathematical models with fractional dynamics resulting in optimal control theory [7]. It is important to point out that some of these quantities are fractional forms of Hamiltonians [5], whence a natural direction of investigation in scientific computing is the design of new computational techniques that preserve the relevant quantities of a physical system described by fractional partial differential equations. It is worth pointing out that this task has been accomplished recently for fractional hyperbolic partial differential equations that extend the well known sine–Gordon and nonlinear Klein–Gordon models from relativistic quantum mechanics, which are models for which a Hamiltonian function exists [8].

The literature also has reports of methods for fractional partial differential equations that do not necessarily preserve the structure of the solutions, but most of the methods proposed are numerically efficient techniques. For example, some highly accurate numerical schemes have been proposed for multi-dimensional space variable-order fractional Schrödinger equations [9] and some techniques have been used to approximate the solutions of Riesz fractional advection–dispersion equations [10]. Other approximation methods based on Legendre polynomials have been designed to solve the fractional two-dimensional heat conduction equation [11], to approximate the solutions of the multi-term time-fractional wave-diffusion equation [12], to solve numerically the two-dimensional variable-order fractional percolation equation in non-homogeneous porous media [13], to estimate the solutions of (3+1)-dimensional generalized fractional KdV–Zakharov–Kuznetsov equations through an improved fractional sub-equation method [14] and to solve fractional sub-diffusion equations with variable coefficients [15]. As a conclusion, many reports show that the development of numerical techniques to solve fractional partial differential equations has been a fruitful avenue of research, but few reports have striven to design structure-preserving techniques for those systems.

In this work, the notion of 'structure preservation' not only refers to the capability of numerical methods to preserve analogues of physical quantities. More generally, these concepts also refer to the capacity of a computational technique to preserve mathematical features of the relevant solutions of continuous systems. Such features may naturally arise from the physical context of the problem. A typical example is the condition of positivity (or non-negativity) of the solutions, which is a natural requirement for problems in which the variables of interest are measured in absolute scales [16]. Other characteristics include the boundedness [17], the monotonicity [18] and the convexity of approximations [19]. In the present work, we will consider an initial–boundary-value problem governed by a multidimensional parabolic equation with Riesz fractional diffusion and convection. The problem is a generalization of various equations from mathematical physics, including the well known Burgers–Fisher and the Burgers–Huxley models, which are equations for which there exist positive and bounded solutions under suitable conditions. In this manuscript, we will propose a structure-preserving and numerically efficient technique to approximate the solutions of that model using fractional centered differences.

This manuscript is organized as follows. In Section 2, we present the relevant definitions on fractional partial derivatives in the Riesz sense, and we introduce the model under investigation. The initial–boundary-value problem of interest is governed