ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A multi-grid technique for coupling fluid flow with porous media flow

Liyun Zuo [a,*], Guangzhi Du [b]

[a] School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, China
[b] School of Mathematics and Statistics, Shandong Normal University, Jinan, Shandong 250014, China

ARTICLE INFO

Article history:
Received 31 July 2017
Received in revised form 19 October 2017
Accepted 3 March 2018
Available online xxxx

Keywords:
Stokes equations
Darcy's law
Beavers–Joseph interface condition
Multi-grid technique
Finite element method

ABSTRACT

In this paper, we consider the coupling of fluid flow with porous media flow. A multi-grid finite element method for the coupled Stokes–Darcy problem with the Beavers–Joseph interface condition is proposed and discussed. The optimal error estimates are obtained. Numerical experiment is given to verify the theoretical analysis and indicate the accuracy and efficiency of the multi-grid method.

1. Introduction

The coupling of fluid flow (governed by the Stokes equations or Navier–Stokes equations) and porous media flow (governed by the Darcy's law) through certain interface conditions (Beavers–Joseph–Saffman or Beavers–Joseph interface conditions) has received great attention and has become a very active research area in recent years. The major reason lies in its wide spectrum of real world applications, such as surface water and groundwater flows, oil flows in a vuggy porous medium, and so on.

Due to the important applications of such coupled problem, many experts devote themselves to studying its mathematical analysis and numerical methods, such as domain decomposition method [1–8], two-grid and multi-grid methods [9–16], local and parallel methods [17,18], temporal extrapolation methods [19–21], stabilized finite element methods [22–24], discontinuous finite element methods [25,26], Lagrange multiplier [27], finite volume method [28], least squares pseudo-spectral method [29], and so on. Moreover, some experts considered the posteriori error estimate [30] and analyzed it for the Stokes–Darcy problems [31,32]. Most of previous works on the coupled Stokes–Darcy problem considered the Beavers–Joseph–Saffman interface condition [33], which neglects the contributions made by the flow in the porous medium region flow to the coupling of the two models and hence the corresponding well-posedness can be demonstrated in a fairly straightforward manner. However, the neglected contributions may be important in some cases, such as groundwater system in karst aquifers.

In this paper, we consider the coupled Stokes–Darcy problem with the Beavers–Joseph interface condition [34], which is more physical and more accurate than the Beavers–Joseph–Saffman interface condition, because it fully accounts for the contributions of two flows at the interface. However, the mentioned coupled problem leads to a lot of difficulties in theoretical analysis and numerical implementation. In fact, theoretical difficulties mainly lies in the indefiniteness of the

* Corresponding author.
E-mail addresses: yeziliyun@126.com (L. Zuo), guangzhidu@gmail.com (G. Du).

https://doi.org/10.1016/j.camwa.2018.03.010
0898-1221/© 2018 Elsevier Ltd. All rights reserved.

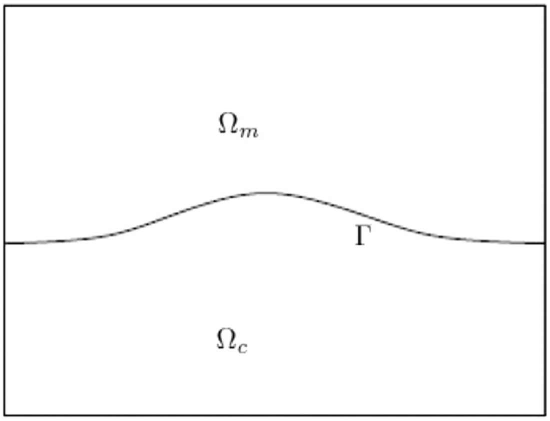

Fig. 1. A global domain Ω consisting of a matrix region Ω_m and a conduit region Ω_c separated by an interface Γ.

bilinear form, Cao and the co-workers in [35] get the well-posedness of the corresponding problem when α is sufficiently small. Then some numerical methods have been developed for solving the coupled Stokes–Darcy problem with the Beavers–Joseph interface condition, such as domain decomposition methods [2,6], two-grid method [12], partitioned time stepping method [20], and so on.

Due to the wide and important applications of the coupled flow problem, the accuracy of Beavers–Joseph interface condition and the few numerical methods for the coupled problem with the Beavers–Joseph interface condition, it is still very necessary to find a simpler, more accurate and more efficient method to solve this coupled problem. We are interested in the two-grid [12,15,36] and multi-grid methods [10,37]. Specially, Mu and Xu in [15] proposed a two-grid scheme, and then Cai and Mu in [10] extended this scheme to multi-grid scheme. However, they all considered the coupled Stokes–Darcy problem with the Beaver–Joseph–Saffman interface condition, and did not obtain the optimal error estimates for the velocity and pressure in the fluid flow region. Later on, a modified two-grid scheme was proposed in [12] for the coupled Stokes–Darcy problem with the Beaver–Joseph interface condition and the optimal error estimates were obtained.

Consequently, we extend the two-grid scheme in [12] to the multi-grid scheme, and propose a multi-grid finite element method for the coupled Stokes–Darcy problem with the Beavers–Joseph interface condition. By this method, the global problem only need to be solved on a very coarse initial grid, and Stokes and Darcy subproblems are solved on a series of refined grids. Theoretical analysis and numerical experiment all demonstrate that the proposed multi-grid method is accurate and very efficient.

The rest of the paper is organized as follows. The coupled Stokes–Darcy problem with Beavers–Joseph interface condition is given in the next section. A multi-grid method is presented in Section 3. In Section 4, convergence of the proposed method is analyzed, followed by the numerical experiment to verify the accuracy and efficiency of the multi-grid method in Section 5. Finally, a conclusion is drawn in Section 6.

2. Coupled Stokes–Darcy problem

Let us consider a coupled Stokes–Darcy problem in a bounded domain $\Omega \subset \mathbb{R}^d (d = 2 \text{ or } 3)$, which consists of a conduit region Ω_c and a matrix region Ω_m, with the interface $\Gamma = \partial\Omega_c \cap \partial\Omega_m$, see Fig. 1.

In the matrix region Ω_m, the flow is governed by the Darcy equations:

$$u_m = -\mathbb{K}\nabla\varphi, \tag{2.1}$$

$$\nabla \cdot u_m = f_2, \tag{2.2}$$

where u_m is the specific discharge, f_2 is the source term. $\mathbb{K}$ is the hydraulic conductivity tensor of the porous media, which denotes permeability of the rock and is assumed to be symmetric and positive definite. For the sake of simplicity, we assume $\mathbb{K}$ is a constant diagonal matrix, i.e. $\mathbb{K} = k\mathbb{I}$, where k is a constant and $\mathbb{I}$ is the identity matrix. φ denotes the piezometric head, which is defined as $z_m + \frac{p_m}{\rho g}$. Here p_m is the dynamic pressure, ρ is the density, g is the gravitational acceleration, z_m is the relative depth from an arbitrary fixed reference height. Without loss of generality, we assume $z_m = 0$.

Eliminating u_m from (2.1)–(2.2), we have the following equation:

$$-\nabla \cdot (\mathbb{K}\nabla\varphi) = f_2. \tag{2.3}$$

In the conduit region Ω_c, the flow is governed by the Stokes equations:

$$-\nabla \cdot \mathbb{T}(u, p) = f_1, \tag{2.4}$$

$$\nabla \cdot u = 0, \tag{2.5}$$