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Asymptotic nonlinear and dispersive pulsatile flow in elastic vessels with cylindrical symmetry

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ABSTRACT

The asymptotic derivation of a new family of one-dimensional, weakly nonlinear and weakly dispersive equations that model the flow of an ideal fluid in an elastic vessel is presented. Dissipative effects due to the viscous nature of the fluid are also taken into account. The new models validate by asymptotic reasoning other non-dispersive systems of equations that are commonly used, and improve other nonlinear and dispersive mathematical models derived to describe the blood flow in elastic vessels. The new systems are studied analytically in terms of their basic characteristic properties such as the linear dispersion characteristics, symmetries, conservation laws and solitary waves. Unidirectional model equations are also derived and analysed in the case of vessels of constant radius. The capacity of the models to be used in practical problems is being demonstrated by employing a particular system with favourable properties to study the blood flow in a large artery. Two different cases are considered: A vessel with constant radius and a tapered vessel. Significant changes in the flow can be observed in the case of the tapered vessel.

1. Introduction

The study of the axisymmetric flow of an inviscid fluid in elastic vessels is important on several accounts but especially because of its applications to the blood flow in arteries. The mathematical modelling of arterial systems is based on the equations of continuum mechanics for the flow of an incompressible fluid in vessels known as the Navier–Stokes equations, [1]. The incompressible Navier–Stokes formulation of the blood flow has the advantage of taking into account the dissipative effects of the flow due to viscosity. On the other hand, the flow exhibits a rather complex structure due to the mechanical interaction between the fluid and the vessel walls. Another very important factor that influences the blood flow is the viscoelastic nature of vessel walls. For example, large arteries deform under blood pressure and they are capable of storing elastic energy during the systolic phase and release it during the diastolic phase. Modelling the elastic properties of the vessels appears to have significant difficulties of mathematical and numerical nature, [1–6].

Several attempts to simplify the study of the blood flow have been made, especially in the case of large vessels with elastic wall that are capable to deform under pressure. In many recent studies the viscosity of the flow has been neglected since otherwise the mathematical modelling becomes very complicated. To this end, the focus is mainly on the inviscid, incompressible and radially symmetric fluid flow equations known to as the Euler equations. These equations written in

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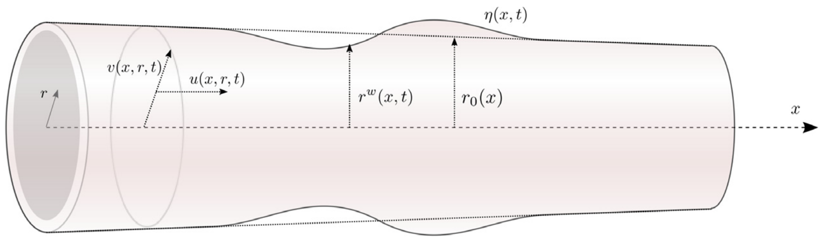

Fig. 1. Sketch of the physical domain for a single vessel segment with elastic and impenetrable wall.

cylindrical coordinates take the form:

$$u_t + uu_x + vu_r + \frac{1}{\rho}p_x = 0\,, \tag{1.1}$$

$$v_t + uv_x + vv_r + \frac{1}{\rho}p_r = 0\,, \tag{1.2}$$

$$u_x + v_r + \frac{1}{r}v = 0\,, \tag{1.3}$$

where $u = u(x, r, t)$, $v = v(x, r, t)$ are the horizontal and radial velocity of fluid respectively, $p = p(x, r, t)$ is the pressure of the fluid, while ρ is the constant density of the fluid.

A sketch of the physical domain of this problem is presented in Fig. 1, where the distance of vessel's wall from the centre of the vessel in a cross section is denoted by $r^w(x, t)$ and depends on x and t while the radius of the vessel at rest is the function $r_0(x)$. In general the deformation of the wall will be a function of x and t. If we denote the radial displacement of the wall by $\eta(x, t)$ then the vessel wall radius can be written as $r^w(x, t) = r_0(x) + \eta(x, t)$.

The governing equations (1.1)–(1.3) combined with initial and boundary conditions form a closed system. A compatibility condition is also applied at the centre of the vessel (due to cylindrical symmetry). Specifically, we assume that

$$v(x, r, t) = 0, \text{ for } r = 0\,. \tag{1.4}$$

On the vessel wall the impermeability condition can be written in the form:

$$v(x, r, t) = \eta_t(x, t) + (r_0(x) + \eta(x, t))_x u(x, r, t), \text{ for } r = r^w(x, t)\,, \tag{1.5}$$

and expresses that the fluid velocity equals the wall speed $v = r^w_t$. The second boundary condition is actually Newton's second law on the vessel wall written in the form:

$$\rho^w h\eta_{tt}(x, t) = p^w(x, t) - \frac{E_\sigma h}{r_0^2(x)}\eta(x, t)\,, \tag{1.6}$$

where ρ^w is the wall density, p^w is the transmural pressure, h is the thickness of the vessel wall, $E_\sigma = E/(1 - \sigma^2)$ where E is the Young modulus of elasticity with σ denoting the Poisson ratio of the elastic wall. In this study we assume that E is a constant and in general we will replace in the notation E_σ by E. It is noted that because the flow is pressure-driven the effect of gravity is neglected. For more information about the derivation of the Euler equations and the boundary conditions we refer to [7,2]. It is noted that assuming a laminar flow and small viscosity the Navier–Stokes equations can be reduced to a modified system which is very similar to the Euler equations, [6], and therefore an analysis on the Euler equations can easily be generalised to the specific simplified viscous case.

Due to the complexity of the Euler equations several one-dimensional models have been introduced, [8–15]. The models include unidirectional, cf. e.g. [16–20], and bidirectional models, cf. e.g. [21–23]. Although these models usually are neither asymptotic models nor dispersive, systematic comparisons between one and three dimensional idealised arterial blood flow models showed a very good agreement, cf. e.g. [24]. Moreover, one-dimensional models can also be used to compute inflow boundary conditions to three-dimensional models. However, one-dimensional models cannot handle curved vessels unless the central axis is a graph of a function. For this reason three-dimensional models cannot be totally replaced by the simple one-dimensional models, cf. e.g. [24–28]. It is noted that the literature is not limited in the above references but is very extensive and we apologise if we do not include the complete literature in the field.

In this paper we derive some new asymptotic one-dimensional model equations of Boussinesq type (weakly non-linear and weakly dispersive) that approximate the system (1.1)–(1.3) with boundary conditions (1.4)–(1.6). The new systems describe inviscid and irrotational fluid flow in elastic vessels of variable diameter and can be used as an alternative to the Euler equations (1.1)–(1.3). We also derive dissipative Boussinesq equations from the Navier–Stokes equations using standard arguments on the velocity profile, [6], and extending the asymptotic reasoning of the inviscid case to the viscous case. The new models are generic and can be used to study the blood flow in large arteries while discarding the dispersive