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Three time integration methods for incompressible flows with discontinuous Galerkin Boltzmann method

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ABSTRACT

This paper presents three time integration methods for incompressible flows with finite element method in solving the lattice-BGK Boltzmann equation. The space discretization is performed using nodal discontinuous Galerkin method, which employs unstructured meshes with triangular elements and high order approximation degrees. The time discretization is performed using three different kinds of time integration methods, namely, direct, decoupling and splitting. From the storage cost, temporal accuracy, numerical stability and time consumption, we systematically compare three time integration methods. Then benchmark fluid flow simulations are performed to highlight efficient time integration methods. Numerical results are in good agreement with others or exact solutions.

1. Introduction

Boltzmann equation with the Bhatnagar–Gross–Krook (BGK) approximation [1] of the collision term has been widely investigated to compute flow and noise. With a suitable grid in the velocity space, the so called standard lattice Boltzmann (SLB) method strongly couples the time and space discretizations along characteristics. Although SLB is easy to implement for parallel computation, it is limited to regular structured meshes and low Reynolds number. To overcome this limitation, finite difference [2–5], finite volume [6–9] and finite element [10–17] methods are used to directly solve the lattice-BGK Boltzmann equation.

Among these approaches, the discontinuous Galerkin (DG) method has been popular because of its geometric flexibility and high order spatial accuracy. After the DG space discretization of the BGK-lattice Boltzmann equation, one can obtain the following semi-discrete ordinary differential equation

$$\underbrace{\mathbf{M}}_{\text{Mass}}\frac{d\boldsymbol{f}}{dt} + \underbrace{\underbrace{\mathbf{S}}_{\text{Stiffness}}\boldsymbol{f} + \underbrace{\mathbf{R}(t,\boldsymbol{f})}_{\text{SI}}}_{\text{convection}} = \mathbf{M}\underbrace{\frac{\mathbf{Q}(\boldsymbol{f})}{\varepsilon}}_{\text{BGK collision}}, \tag{1}$$

where $\boldsymbol{f}$ is the set of particle distribution functions, SI means surface integration generated by information transmission between adjacent elements, and ε is the relaxation time. Actually, in most conditions time discretization algorithms such as linear multi-step and explicit Runge–Kutta schemes can be applied to Eq. (1). Shi et al. [12] used a third order Adams–Bashforth scheme to simulate steady flow around a single cylinder. But, when the Reynolds number increases, the relaxation time decreases, leading to a significantly stiff collision term. For explicit time integration, very small time step should be

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used to satisfy numerical stability. Naturally one would resort to implicit integration techniques. Due to prohibitive cost required for the inversion of the convection and collision terms, in most applications fully implicit integration algorithm presents considerable limitations. A cost effective way is to implement a semi-implicit scheme on the collision term. Min and Lee [15] decoupled the collision step from the streaming step, and transformed the implicit collision step into an explicit step based on the conservation of moments. Although fourth order Runge–Kutta scheme [15,17] or exponential time marching scheme [16] is applied to the streaming step, it is second order temporal accuracy in whole.

On the other hand, implicit–explicit (IMEX) Runge–Kutta schemes [18] can be used to hyperbolic systems with stiff terms. The convection term is treated explicitly while the BGK collision term is treated implicitly. Moreover, the IMEX Runge–Kutta schemes require conservative and non-oscillatory space discretizations. Finite difference or finite volume space discretizations [19] such as high order WENO schemes [20] can be used. For the DG space discretization, the convection term does not satisfy exact conservation of macroscopic variables at the discrete level.

Besides direct solution of the lattice-BGK Boltzmann equation, by means of the Strang splitting method [21] the convection and collision terms can be treated separately. Then the implicit time integration can be applied to the partial differential equation containing the collision term. This time-splitting technique presents similar properties to decouple the collision step from the streaming step.

More than the collision term, the convection term may generate large stiffness. For small geometrical features or *hp* refinement in DG space discretization, it demands very small time step. Considering the enormously expensive solution of large systems of equations, one would not adopt an implicit solver for the convection term. When the explicit time discretization is used, some properties like strong stability preserving (SSP) [22,23] can be guaranteed to enhance numerical stability.

Even though various time integration methods were developed, systematic comparison of them has not been made so far. Possibly these time discretization schemes were designed for certain kinds of problems. Therefore, in this paper we compare three time integration methods to obtain an optimal scheme for incompressible flows. The underlying equation and the space discretization are specified as the lattice-BGK Boltzmann equation and the nodal DG method, respectively. Due to huge time consumption in practical computational fluid dynamic (CFD) optimization, it is worth finding accurate and cost effective approaches for the CFD solver.

The rest of the paper is organized as follows. In Section 2, we describe the underlying equation, the nodal DG space discretization and the boundary conditions. In Section 3, three time integration methods are briefly introduced. In Section 4, we compare the time integration methods through numerical tests in detail. Finally, in Section 5, the conclusion remarks are provided.

2. Pre time integration

2.1. The underlying equation

The lattice-BGK Boltzmann equation [24] reads:

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{e}_\alpha \cdot \nabla f_\alpha = \frac{f_\alpha^{\mathrm{eq}} - f_\alpha}{\lambda}, \tag{2}$$

where f_α is the particle distribution function at the αth lattice velocity $\boldsymbol{e}_\alpha$, λ is the relaxation time towards equilibrium distribution function f_α^{eq}. Here, we consider the commonly used D2Q9 lattice model given as

$$\boldsymbol{e}_\alpha = \begin{cases} (0,0) & \alpha = \{0\} \\ c\,(\cos\theta_\alpha, \sin\theta_\alpha) & \alpha = \{1,2,3,4\}, 2\theta_\alpha + \pi = \alpha\pi \\ \sqrt{2}c\,(\cos\theta_\alpha, \sin\theta_\alpha) & \alpha = \{5,6,7,8\}, 4\theta_\alpha + \pi = 2\alpha\pi, \end{cases}$$

where c is the characteristic lattice velocity. The equilibrium distribution function is given as

$$f_\alpha^{\mathrm{eq}} = \rho w_\alpha \left[1 + \frac{\boldsymbol{e}_\alpha \cdot \boldsymbol{u}}{c_s^2} + \frac{1}{2}\left(\frac{\boldsymbol{e}_\alpha \cdot \boldsymbol{u}}{c_s^2}\right)^2 - \frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2c_s^2} \right], \tag{3}$$

where ρ and $\boldsymbol{u}$ are the macroscopic density and velocity, respectively, c_s is the speed of sound, and $w_0 = 4/9$, $w_{1\text{–}4} = 1/9$, and $w_{5\text{–}8} = 1/36$ are the lattice weights. In this model, the relaxation time has a relation with the kinematic viscosity $\nu = \lambda {c_s}^2$. Through Chapman–Enskog asymptotic analysis, the macroscopic pressure is given as $p = \rho {c_s}^2$. Taking the moments of particle distribution functions, we obtain

$$\rho = \sum_{\alpha=0}^{8} f_\alpha, \quad \rho\boldsymbol{u} = \sum_{\alpha=0}^{8} \boldsymbol{e}_\alpha f_\alpha. \tag{4}$$

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