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Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Uniqueness and numerical scheme for the Robin coefficient identification of the time-fractional diffusion equation

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ARTICLE INFO

Article history:
Received 25 January 2017
Received in revised form 21 February 2018
Accepted 10 March 2018
Available online xxxx

Keywords:
Inverse problem
Robin coefficient
Fractional diffusion equation
Uniqueness

ABSTRACT

We study an inverse problem of determining the Robin coefficient of fractional diffusion equation from a nonlocal boundary condition. Based on the property of Caputo fractional derivative, the uniqueness is proved. The numerical schemes for the direct problem and the inverse problem are developed. Three examples are given to show the effectiveness of the presented methods.

1. Introduction

Let Ω be a bounded domain in R^N with a Lipschitz boundary Γ which contains three nonoverlapping parts, $\Gamma = \Gamma_{Rob} \bigcup \Gamma_{Dir} \bigcup \Gamma_{Neu}$. We consider the following inverse Robin coefficient problem for a fractional diffusion equation:

$$\begin{cases} \partial_t^\alpha u(x,t) = \Delta u(x,t) + f(x,t), & x \in \Omega, t \in (0,T), 0 < \alpha < 1, \\ u(x,t) = 0, & x \in \Gamma_{Dir}, t \in [0,T], \\ -\dfrac{\partial u(x,t)}{\partial n} = g(x,t), & x \in \Gamma_{Neu}, t \in [0,T], \\ -\dfrac{\partial u(x,t)}{\partial n} = \lambda(t) u(x,t) + h(x,t), & x \in \Gamma_{Rob}, t \in [0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where n is the outward normal vector on Γ, the measure of Γ_{Rob} is not zero, i.e. $meas(\Gamma_{Rob}) \neq 0$, ∂_t^α is the Caputo fractional derivative of order α defined by

$$\partial_t^\alpha u(x,t) = I_t^{1-\alpha} u_t(x,t), \quad 0 < \alpha < 1,$$

where the fractional integral of order α is defined by

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau,$$

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https://doi.org/10.1016/j.camwa.2018.03.017

and $f(x,t)$, $g(x,t)$, $h(x,t)$, $u_0(x)$ are given functions, $\lambda(t) \in \mathcal{S} = \{\lambda(t) \in L^\infty(0,T)|\lambda(t) \geq \lambda_0 > 0\}$ is a Robin coefficient describing the convection between the solute in a body and one in the ambient environment. If $\lambda(t)$ is given, the problem (1.1) is a well-posed direct problem, see [1] for the existence and uniqueness of a weak solution. As for the high dimensional case, the existence and uniqueness of classical solution have been studied in [2].

When $\lambda(t)$ is unknown and need to be recovered, we should add some over-specified condition. There exist two classes of the over-specified conditions for Robin coefficient identification. One is the Dirichlet condition on part of Γ_{Rob} [1], and the other is a nonlocal boundary condition on Γ_{Rob} [3].

For $\alpha = 1$ with the Dirichlet over-specified condition, the inverse Robin coefficient problem has been widely studied. One can refer to [4–6] for the numerical methods to solve this problem. With the over-specified nonlocal boundary condition, Slodicka and Van Keer [7] studied the uniqueness of the Robin coefficient identification in a semilinear parabolic equation, and proposed a time discretization with some convergence analysis. In 2010, they [8] extended the analysis to estimating a temporally-dependent Robin coefficient in a nonlinear boundary condition for one-dimensional heat equation, and showed the existence and uniqueness of the solution.

For $0 < \alpha < 1$ with the Dirichlet over-specified condition, Wei and Zhang [9] transformed the identification problem into a nonlinear Volterra integral equation and then solved the first kind integral equation with boundary element method combined with conjugate gradient method. In [1], Wei and Wang reduced the inverse problem into a variational problem and deduced the gradient of the regularization functional based on an adjoint problem.

To our knowledge, there are no literatures to give the uniqueness under either of the over-specified condition for $0 < \alpha < 1$. Motivated by [7], we will prove the uniqueness of the inverse problem of determining the Robin coefficient of fractional diffusion equation from a nonlocal boundary condition.

The paper is organized as follows. In Section 2, we present some preliminaries and prove the uniqueness of the Robin coefficient identification. The numerical implementation methods for the direct problem and the inverse problem are given in Section 3. In Section 4, we show some numerical results for three examples to illustrate the effectiveness of the methods developed in Section 3. Finally, concluding remarks are given in Section 5.

2. Uniqueness

In this section, using the formula for fractional derivative of the product of two functions given by Alikhanov [10], we will give a lower bound of the inner product of a function and its fractional derivative.

Lemma 2.1 (*[11]*). *If $\alpha, \beta > 0$, then the equation*

$$(I_t^\alpha I_t^\beta f)(t) = (I_t^{\alpha+\beta} f)(t) \tag{2.2}$$

is satisfied at almost every point $t \in [0,T]$ for $f(t) \in L^1(0,T)$.

Lemma 2.2 (*[10]*). *For any functions $v(t) \in AC[0,T]$, i.e., $v(t)$ is absolutely continuous on $[0,T]$, the following equality takes place:*

$$v(t)\partial_t^\alpha v(t) = \frac{1}{2}\partial_t^\alpha v^2(t) + \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^1 \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi \frac{v'(\eta)d\eta}{(t-\eta)^\alpha}\right)^2,$$

where $0 < \alpha < 1$.

Corollary 2.1. *If $u(x,t) \in AC[0,T]$ for a.e. $x \in \Omega$, we have*

$$\partial_t^\alpha (u(x,t), u(x,t))_\Omega \leq 2(\partial_t^\alpha u(x,t), u(x,t))_\Omega.$$

Proof. We just need to verify that

$$\int_\Omega \partial_t^\alpha u^2(x,t)dx \leq 2\int_\Omega u(x,t)\partial_t^\alpha u(x,t)dx.$$

It is obvious from Lemma 2.2 that $\partial_t^\alpha u^2(x,t) \leq 2u(x,t)\partial_t^\alpha u(x,t)$ which completes the proof. □

Suppose $\lambda \in C[0,T]$, $\lambda(t) \geq 0$. Then the variational formulation of (1.1) is given by: find $u_\lambda \in L^2((0,T), H^1(\Omega))$, with $\partial_t^\alpha u_\lambda \in L^2((0,T), L^2(\Omega))$, such that

$$(\partial_t^\alpha u_\lambda, \varphi) + (\nabla u_\lambda, \nabla\varphi) + \lambda(u_\lambda, \varphi)_{\Gamma_{Rob}} + (h, \varphi)_{\Gamma_{Rob}} + (g, \varphi)_{\Gamma_{Neu}} = (f, \varphi) \tag{2.3}$$

for any $\varphi \in V = \{\varphi \in H^1(\Omega); \varphi = 0 \text{ on } \Gamma_{Dir}\}$.

There are two kinds of methods to obtain the existence and uniqueness of the weak solution of (1.1) [1,12]. Here we use the method in Section 3 in [1]. We integrate (2.3) over $[0,T]$ and denote $I = [0,T]$, $\Lambda = \Omega \times [0,T]$, $\Lambda_{Rob} = \Gamma_{Rob} \times [0,T]$, $\Lambda_{Neu} = \Gamma_{Neu} \times [0,T]$, then we get

$$(\partial_t^\alpha u_\lambda, \varphi)_\Lambda + (\nabla u_\lambda, \nabla\varphi)_\Lambda + (\lambda u_\lambda, \varphi)_{\Lambda_{Rob}} = (f, \varphi)_\Lambda - (h, \varphi)_{\Lambda_{Rob}} - (g, \varphi)_{\Lambda_{Neu}}, \tag{2.4}$$

Please cite this article in press as: J.-G. Wang, et al., Uniqueness and numerical scheme for the Robin coefficient identification of the time-fractional diffusion equation, Computers and Mathematics with Applications (2018), https://doi.org/10.1016/j.camwa.2018.03.017.