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A space–time spectral collocation method for the two-dimensional variable-order fractional percolation equations

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ABSTRACT

In this article, we introduce a space–time spectral collocation method for solving the two-dimensional variable-order fractional percolation equations. The method is based on a Legendre–Gauss–Lobatto (LGL) spectral collocation method for discretizing spatial and the spectral collocation method for the time integration of the resulting linear first-order system of ordinary differential equation. Optimal priori error estimates in L^2 norms for the semi-discrete and full-discrete formulation are derived. The method has spectral accuracy in both space and time. Numerical results confirm the exponential convergence of the proposed method in both space and time.

1. Introduction

In this paper, we introduce and analyze a space–time spectral collocation method [1] for the following two-dimensional variable-order fractional percolation equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(A(x,y)\frac{\partial^{\alpha(x,y)}u}{\partial x^{\alpha(x,y)}}) + \frac{\partial}{\partial y}(B(x,y)\frac{\partial^{\beta(x,y)}u}{\partial y^{\beta(x,y)}}) + f(x,y,t), \tag{1}$$

with initial condition

$$u(x,y,t_0) = \varphi(x,y)\ (x,y) \in \Omega = (a_1,a_2)\times(b_1,b_2), \tag{2}$$

and Dirichlet boundary conditions

$$\begin{aligned} &u(a_1,y,t) = 0,\ u(a_2,y,t) = \Psi_1(y,t),\ y\in(b_1,b_2),\ t\in J=(t_0,T),\\ &u(x,b_1,t) = 0,\ u(x,b_2,t) = \Psi_2(x,t),\ x\in(a_1,a_2),\ t\in J, \end{aligned} \tag{3}$$

where $(x,y,t) \in Q_T = [a_1,a_2]\times[b_1,b_2]\times[t_0,T]$, $\{\Psi_i\}_{i=1}^2$, $f(x,y,t)$, $\varphi(x,y)$ are all known functions, and $0 < \alpha(x,y),\ \beta(x,y) < 1$.

Here, as usual, the Riemann–Liouville fractional derivative is defined as (see [2])

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_a^x \frac{u(s,y,t)}{(x-s)^\alpha}ds,$$

where $0 < \alpha < 1$.

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Seepage flow problems have been widely discussed in many research fields, such as seepage hydraulics, groundwater hydraulics, groundwater dynamics and fluid dynamics in porous media [3–5]. The traditional partial differential equations for single phase isothermal seepage flow under the hypotheses of continuity and Darcy's law can be written as

$$\frac{\partial}{\partial x}(A\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(B\frac{\partial u}{\partial y}) + h(x, y, t) = \frac{1}{v}\frac{\partial u}{\partial t}, \tag{4}$$

where $(x, y) \in \Omega$, A and B are the percolation coefficients along the x and y direction, respectively; u is the pressure; v is velocity, $h = h(x, y, t)$ is the source term; and Ω denotes the percolation domain. The equation in non-homogeneous porous media can be written as follows:

$$\frac{\partial}{\partial x}(A(x, y)\frac{\partial^{\alpha(x,y)}u}{\partial x^{\alpha(x,y)}}) + \frac{\partial}{\partial y}(B(x, y)\frac{\partial^{\beta(x,y)}u}{\partial y^{\beta(x,y)}}) + f(x, y, t) = \frac{\partial u}{\partial t}, \tag{5}$$

where $(x, y) \in \Omega,\ A(x, y) = vA,\ B(x, y) = vB,\ f(x, y, t) = vh(x, y, t)$. The above equation forms the focus of this paper and it is known as the two-dimensional variable order fractional percolation equation.

The two-dimensional variable order fractional percolation equations are always used to describe physical process and have attracted attention of numerical researchers. Over the past years, several analytic methods and numerical schemes have been developed for solving the space or time fractional partial differential equations. For instance, finite difference approximations were presented for the fractional Fokker–Planck equation [6], the variable-order nonlinear fractional diffusion equation [7] and the two-sided space-fractional partial differential equation [8]. Chen [9] proposed a novel implicit finite difference method for the three-dimensional fractional percolation equation. Finite element method was proposed for fractional order differential equation with constant coefficients in the fractional derivative terms [10,11]. Zhuang [12] and Liu [13] presented numerical analytical method for the anomalous subdiffusion equation. Recently, Bhrawy [14] developed an improved collocation method for multi-dimensional space–time variable-order fractional Schrödinger equations. Numerical methods were considered for a two-dimensional variable-order modified diffusion equations [15] and a new space–time variable fractional order advection–dispersion equation [16]. Methods for the multi-term time-fractional diffusion-wave equations include analytical solution [17] and finite difference approximation [18]. Existing methods for the two-dimensional variable order fractional percolation equations with variable coefficients only have an implicit alternating direct method [19].

In this paper, we propose a numerical scheme for solving Eq. (1). We apply spectral collocation method [20–22] for discretizing spatial derivatives and time derivatives, which is high-order accurate in both space and time. The main advantage of spectral methods is their superior accuracy for problems whose solutions are sufficiently smooth functions. Through being compared to the implicit alternating direct method [19], the numerical results show the fast exponential convergence.

The outline of this paper is as follows. A spectral collocation method for nonlinear initial value problem is introduced in Section 2. In Section 3, we describe the LGL spectral method for two-dimensional variable-order fractional percolation equation in space. In Section 4, L^2 error estimating for the semi-discrete LGL spectral method is constructed. The analysis of the full discrete scheme is given in Section 5. Section 6 provides extensive numerical results to assess the convergence and accuracy of the method. Finally, Section 7 ends this paper with a brief conclusion.

2. The spectral collocation method

In this section, we briefly introduce the spectral collocation method for solving nonlinear initial-value problems. In $\Lambda = [-1, 1]$, Legendre–Gauss–Lobatto (LGL) points $\{\widetilde{x}_j\}_{j=0}^N$ are roots of $\partial_{\widetilde{x}} L_N(\widetilde{x})$, where $\widetilde{x}_0 = -1,\ \widetilde{x}_N = 1,\ L_N(\widetilde{x})$ is the Legendre polynomial [12].

At the same collocation points, we differentiate the polynomial and evaluate the polynomial. Noting $f_k = f(\widetilde{x}_k)$, $\widetilde{F}_N(\widetilde{x})$ is a function associated with $\widetilde{x}$, then

$$\widetilde{F}_N(\widetilde{x}) = \sum_{k=0}^{N} f_k \widetilde{L}_k(\widetilde{x}), \tag{6}$$

where $\widetilde{L}_k(\widetilde{x})$ are Lagrange interpolation polynomials, that is

$$\widetilde{L}_k(\widetilde{x}) = \prod_{j=0, j\neq k}^{N} \frac{\widetilde{x} - \widetilde{x}_j}{\widetilde{x}_k - \widetilde{x}_j} = \frac{1}{N(N+1)L_N(\widetilde{x}_k)} \frac{(\widetilde{x}^2 - 1)L'_N(\widetilde{x})}{\widetilde{x} - \widetilde{x}_k}, \quad k = 0, 1, \ldots, N,$$

satisfying

$$\widetilde{L}_k(\widetilde{x}_j) = \begin{cases} 0, & \text{if } j \neq k, \\ 1, & \text{if } j = k. \end{cases}$$