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Time second-order finite difference/finite element algorithm for nonlinear time-fractional diffusion problem with fourth-order derivative term ☆

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ABSTRACT

In this article, we study and analyze a Galerkin mixed finite element (MFE) method combined with time second-order discrete scheme for solving nonlinear time fractional diffusion equation with fourth-order derivative term. We firstly introduce an auxiliary variable $\sigma = \triangle u$, reduce the fourth-order problem into a coupled system with two equations, discretize the obtained coupled system at time $t_{k-\frac{\alpha}{2}}$ by a second-order difference scheme with second-order approximation for fractional derivative, then formulate mixed weak formulation and fully discrete MFE scheme. Further, we give the detailed proof for stability of scheme, the existence and uniqueness of MFE solution, and a priori error estimates. Finally, by some numerical computations, we test the theoretical results, which illustrate that we can obtain the numerical results for two variables, moreover, we arrive at second-order time convergence orders, which are higher than the ones yielded by the $L1$-approximation.

1. Introduction

Fractional differential equations (FDEs) including many class of types play a very important role in many fields of engineering and sciences. So, some good mathematical methods for solving analytically or numerically the FDEs should be developed and proposed. However, in view of the existence of complex fractional derivatives, many FDEs cannot be solved by some analytical methods, so the numerical methods need to be studied to look for the numerical solutions. Moreover, FDEs includes many important physical problems such as fractional diffusion equations, fractional water wave problems, fractional fourth-order (diffusion) equations. So far, fractional diffusion equations, which cover time, space and space–time fractional diffusion problems and also include second-order and fourth-order fractional diffusion problems, have been solved by many numerical methods. Now, we give some comments on numerical methods for fractional diffusion problems.

In [1], Li et al. developed numerical methods for nonlinear subdiffusion and superdiffusion problems with space–time fractional derivative. In [2], Quintana-Murillo and Yuste gave the discussions on a finite difference method for fractional diffusion-wave and diffusion problems. In [3], Shivanian solved the 2D convection–diffusion–reaction equations with time fractional derivative by local radial basis function interpolation method. In [4], Aslefallah and Shivanian considered a

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meshless method for solving nonlinear fractional reaction–diffusion equation. In [5], Shen et al. considered a characteristic difference method for advection–diffusion equation including the variable-order fractional derivative. In [6], Sun et al. studied finite difference schemes for time fractional diffusion equation. In [7], Liu et al. discussed and analyzed a finite element approximation with a fast solution technique for the space–time fractional diffusion equation. In [8], Wang and Du provided a fast solution method for solving diffusion equations covering space-fractional derivative. In [9], Zhuang et al. studied some numerical methods for nonlinear fractional advection diffusion problems with the variable-order fractional derivative. In [10], Zhao et al. studied a nonconforming finite element method for time fractional diffusion equations. In [11], Meerschaert and Tadjeran proposed finite difference methods for advection–dispersion flow equations with fractional derivative. In [12], Yang et al. gave a finite volume method for reaction–diffusion problems with space-fractional derivative. In [13], Feng et al. provided a finite volume method for diffusion problem with two-sided space-fractional derivative and discussed the analysis on stability and convergence. In [14], Li and Huang looked for the numerical solution of the 2D time-space fractional diffusion-wave equation by using ADI Galerkin finite element methods. In [15], Jiang and Ma discussed finite element methods for 1D time-fractional diffusion equations. In [16], Ford et al. developed a finite element method for time fractional diffusion problem. In [17], Zhao et al. solved time-space fractional diffusion equation by considering a Galerkin finite element scheme. In [18], Jin et al. studied the Galerkin finite element method for a diffusion problem covering multi-term time-fractional derivative. In [19], Zhang et al. gave the studies of numerical solution for modified fractional diffusion equation by finite difference/element method. In [20], Wang and Vong looked for the numerical solutions for two classes of fractional diffusion equations by considering compact difference schemes. In [21], Bu et al. developed the Galerkin finite element method for diffusion equations with Riesz space fractional derivative. In [22], Zeng et al. solved the time-fractional subdiffusion equation by applying finite difference/element approaches. In [23], Lin and Xu, solved the time-fractional diffusion problem by considering finite difference/spectral approximations. In [24], Heydari et al. developed the wavelets method for solving numerically the time fractional diffusion-wave equation. In [25], Semary et al. solved nonlinear fractional reaction–diffusion models by controlled Picard method.

In this article, we also develop a finite element method to solve the following the nonlinear fourth-order diffusion problems

$$\frac{\partial u}{\partial t} - \frac{\partial^\alpha \triangle u}{\partial t^\alpha} - \triangle u + \triangle^2 u = f(u) + g(\mathbf{x}, t), (\mathbf{x}, t) \in \Omega \times J, \tag{1.1}$$

with boundary condition

$$u(\mathbf{x}, t) = \triangle u(\mathbf{x}, t) = 0, (\mathbf{x}, t) \in \partial\Omega \times \bar{J}, \tag{1.2}$$

and initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \mathbf{x} \in \Omega, \tag{1.3}$$

where $\Omega \subset R^d (d \le 2)$ satisfying *Lipschitz* continuous boundary $\partial\Omega$ is a bounded convex polygonal space domain and $J = (0, T]$ is the time interval satisfying $0 < T < \infty$. $\triangle$ is Laplacian operator, $\frac{\partial^\alpha \triangle u}{\partial t^\alpha}$ is an anomalous diffusion term, which reflects the anomalous diffusion behavior of diffusion processes. $g(\mathbf{x}, t)$ and $u_0(\mathbf{x})$ are given known functions, the nonlinear term $f(u)$ satisfies the assumed conditions: $f(u)$ is the polynomial of u or $|f(u)| \le C|u|$ and $|f'(u)| \le C$, where C are different positive constants and $\frac{\partial^\alpha w(\mathbf{x},t)}{\partial t^\alpha}$ is defined by the following Riemann–Liouville fractional derivative

$$\frac{\partial^\alpha w}{\partial t^\alpha}(\mathbf{x}, t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{w(\mathbf{x}, s)}{(t-s)^\alpha} ds, 0 < \alpha < 1. \tag{1.4}$$

Fourth-order fractional (diffusion) equations have so many applications in many scientific fields, such as pattern formation of bistable systems, traveling waves of reaction–diffusion systems, and propagation of domain walls in liquid crystals. So, more and more researchers have started to consider their solutions by some analytical and numerical methods. In [26], Guo et al. solved some time-fractional fourth-order problems by using fully discrete local discontinuous Galerkin methods. In [27], Wei et al. made the numerical analysis on a local discontinuous Galerkin method for solving fourth-order problems with time-fractional derivative. In [28–30], authors developed some finite element schemes for linear or nonlinear fourth-order FDEs and discussed the numerical theories based on the lower approximate methods for time fractional derivative. In [31], Tariq and Akram considered quintic spline technique for fourth-order PDEs with time fractional derivative. In [32], Zhang and Pu studied a compact difference scheme for a fourth-order sub-diffusion equation with fractional derivative in time. In [33], Ji et al. studied a finite difference method for fourth-order fractional sub-diffusion equations. In [34], Hu and Zhang developed finite difference methods for fractional diffusion-wave and subdiffusion systems with fourth-order derivative.

Recently, Gao et al. [35] developed the finite difference schemes based on a second-order approximate scheme at time $t_{k-\frac{\alpha}{2}}$. Following the idea proposed in [35], Sun et al. [36] gave some second order difference schemes for solving wave equations with fractional derivative. Wang et al. [37] combined second-order time approximation with the finite element method to solve nonlinear fractional Cable equation. In this article, our target is to develop a mixed element method with a derived fractional approximation based on the idea obtained in [35,37] to solve the nonlinear fourth-order time fractional