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Optimal investment problem under non-extensive statistical mechanics

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ABSTRACT

The optimal investment problem in which the asset price process is modeled by the non-extensive statistical mechanics is studied in this paper. By the methods of deterministic control and the dynamic programming, we obtain the optimal strategy with logarithmic utility function, power utility function and quadratic utility function, respectively. Moreover, the numerical results indicate that the optimal investment strategy is affected by the non-extensive parameter, the proportion invested in the risky asset decreases as the wealth increases under quadratic utility function, but it remains unchanged under power utility and logarithmic utility function.

1. Introduction

The optimal consumption and investment problem is an important issue in finance. Given an investment horizon, the objective of an agent is to maximize the expected utility of consumption, the expected utility of the terminal wealth, or some combination of these two quantities. Many researchers are interested in this problem, where Merton [1] has done a ground-breaking job. He [2] first discussed the optimal consumption and investment problem in continuous-time setting. The HJB equations were obtained for the constant coefficient models of power utility functions. The solutions were given, and the solutions of the exponential utility at infinite time were studied. Bismut [3] studied the optimal consumption by the stochastic duality theory. Harrison and Pliska [4] introduced a martingale method to deal with more general price processes, but the explicit solution was given in rare cases. Some papers extended the problem of consumption and investment to this problem with constraints [5–12].

Markowitz's mean–variance model for portfolio selection coincides with a quadratic utility function. Li and Ng [13] studied a multi-period mean–variance optimal portfolio selection problem with embedding technique. Zhou and Li [14] investigated a continuous-time mean–variance optimal portfolio selection problem by stochastic control theory, in which they maximized the expected quadratic utility function. Extensions of this problem can be found in some papers [15–18]. Since then, the literature on dynamic portfolio selection problem can be solved by expected quadratic utility maximization problem.

In above paper, the financial markets were the classical Brownian-Motion-driven model. The assumption that the price of risky assets follow Geometric Brown motion implies that the price change is independent and the distribution of log-returns is normal. However, it is well-known that the distribution of empirical returns do not follow log-normal distribution and has characteristics of fat-tails [19–21]. To further accurately fit the returns of risky assets, several modifications have been developed. One approach is to construct a discontinuous model with Poisson jumps as was done by Merton [22], or

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via a Levy model [23,24]. The other approach is to introduce a stochastic volatility model. The non-extensive statistical mechanics proposed by Tsallis [25] is a stochastic volatility model, which is a generalization of the classical Boltzmann–Gibbs statistics. It was rapidly applied to fit the distribution of the risky assets return. For example, Queirs [26] studied the return distributions for Dow Jones and NYSE and found that the non-extensive statistical mechanics can fit the return distributions well. Kozuki and Fuchikami [27] found that the distributions of foreign exchange rates can be modeled by Tsallis distributions. Borland [28,29] used the non-extensive statistical mechanics to model the changes of stock prices and obtained closed-form solutions for European options. The non-extensive statistical mechanics has also been applied to other financial field. Namaki and Lai et al. [30] applied the Tsallis non-extensive statistical mechanics to detect crises of the financial markets. Zhao and Xiao [31,32] applied the non-extensive statistical methodology to deal with portfolio selection problem.

In this paper, we consider the risky assets price model with the non-extensive statistical mechanics and put forward optimal investment problems with various utility functions.

This paper is organized as follows. Section 2 presents the model with the non-extensive statistical mechanics and proposes the optimal investment problem. Section 3 solves the optimal investment problem with Logarithmic utility by deterministic control and obtains the optimal strategy. Section 4 solves the optimal investment problem with power utility and quadratic utility by HJB equation, respectively. Section 5 discusses the numerical results and Section 6 concludes this paper.

2. The market model and the optimization problem

Suppose there is a financial market with $n+1$ assets on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. One asset is a risk-free asset whose price process $S_0(t)$ satisfies

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, \\ S_0(0) = s_0 > 0. \end{cases} \tag{2.1}$$

The other n assets are risky assets whose prices satisfy

$$\begin{cases} dS_i(t) = S_i(t)\{\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)d\Omega_j(t)\}, \\ S_i(0) = s_i > 0, i = 1, 2, \ldots, n \end{cases} \tag{2.2}$$

where $\mu_i(\cdot)$ and $\sigma_{ij}(\cdot)$ are the appreciation and volatility, respectively. The process $\Omega_j(t)$ is defined by

$$d\Omega_j(t) = P(\Omega_j, t)^{\frac{1-q_j}{2}} dW_j(t) \tag{2.3}$$

where $W(t) = (W_1(t), W_2(t), \ldots, W_n(t))'$ is a R^n-dimensional Brown motion. $P(\Omega_j, t)$ is Tsallis distribution

$$P(\Omega_j, t) = \frac{1}{z(t)}(1 - \beta(t)(1-q_j)\Omega_j^2)^{\frac{1}{1-q_j}} \tag{2.4}$$

with

$$z_j(t) = ((2-q_j)(3-q_j)ct)^{\frac{1}{3-q_j}},$$

$$\beta_j(t) = c_j^{\frac{1-q_j}{3-q_j}}((2-q_j)(3-q_j)t)^{\frac{-2}{3-q_j}},$$

$$c_j = \frac{\pi}{q_j - 1}\frac{\Gamma^2(\frac{1}{q_j-1} - \frac{1}{2})}{\Gamma^2(\frac{1}{q_j-1})}.$$

For $q_j = 1$, $\Omega_j(t)$ reduces the standard Brown motion. The variance of the Tsallis distributions is given by

$$E[\Omega_j^2(t)] = \frac{1}{(5-3q)\beta_j(t)}.$$

Since this expression diverges for $q_j \geq \frac{5}{3}$, we assume $1 \leq q_j < \frac{5}{3}$. For $1 < q_j < \frac{5}{3}$, it exhibits power law tails and has finite variance, which generalizes the standard Brown motion. Hence, this model can more accurately fit the movements of risky assets price.

Consider that an investor with initial wealth x_0 enters the financial market and trades continuously within a time horizon $[0, T]$. The investor is allowed to adjust portfolio at time t. Define the self-financing admissible portfolio process as $\pi(t) = (\pi_1(t), \pi_2(t), \ldots, \pi_n(t))' \in \mathcal{A}(t)$, where $\pi_i(t)$ is the proportion of the investor's wealth invested in the ith risky asset at time t. The corresponding wealth process $X(t)$ is represented by

$$dX(t) = X(t)[(r(t) + \tilde{\mu}(t)'\pi(t))dt + \pi(t)'\sigma(t)P_q dW(t)] \tag{2.5}$$