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The backward problem for a time-fractional diffusion-wave equation in a bounded domain

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ABSTRACT

This paper is devoted to solve the backward problem for a time-fractional diffusion-wave equation in a bounded domain. Based on the series expression of the solution for the direct problem, the backward problem for searching the initial data is converted into solving the Fredholm integral equation of the first kind. The existence, uniqueness and conditional stability for the backward problem are investigated. We use the Tikhonov regularization method to deal with the integral equation and obtain the series expression of the regularized solution for the backward problem. Furthermore, the convergence rate for the regularized solution can be proved by using an a priori regularization parameter choice rule and an a posteriori regularization parameter choice rule. Numerical results for five examples in one-dimensional case and two-dimensional case show that the proposed method is efficient and stable.

1. Introduction

Time-fractional diffusion equations have attracted much more attentions recently because of their successful applications in anomalous diffusion and mechanical fields. They can be used to describe the continuous time random walk phenomenon [1], option pricing [2], superdiffusion and subdiffusion phenomena [3–5]. Due to the memory property of fractional derivatives, time-fractional diffusion equations have advantages in describing hereditary diffusions.

Direct problems for time-fractional diffusion equations have been studied extensively, see [6,7] for examples. The inverse problems for fractional diffusion equations have also been investigated recently, refer to [7–14]. The backward problem of diffusion process aims at detecting the previous status of physical field from its final time information and is of great importance in engineering. For the time-fractional diffusion equations, the backward problems have been studied by many researchers. Sakamoto and Yamamoto [7] proved the uniqueness for the backward problem of a time-fractional subdiffusion equation. Liu et al. [15] solved the backward problem in one-dimensional case by a quasi-reversibility regularization method with some special coefficients. Ren et al. [16] used a truncation regularization method to solve the backward problem in one-dimensional case. Wang et al. in [17] used a Tikhonov method to solve the backward problem in a general bounded domain. Ruan et al. [18] recovered the initial data and the source term simultaneously. The modified quasi-boundary value method [19], the iterative regularization method [20] have also been proposed to solve the backward problem for a time-fractional diffusion equation.

Recently, time-fractional diffusion-wave equations become a new topic in physics and engineering fields. Direct problems for time-fractional diffusion-wave equations have been investigated, such as analytic solutions [21,22], weak solutions [7] and numerical methods [23–25], and [26,27] extend numerical methods to distributed orders. However, as we know, inverse

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problems for fractional diffusion-wave equations have only a few papers such as inverse source problems on an unbounded domain [28] and on a bounded domain [29], the uniqueness of inverse coefficients [30]. In this paper, we focus on a backward problem.

Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficient smooth boundary $\partial\Omega$. In this paper, we consider the following time fractional diffusion-wave problem with homogeneous Dirichlet boundary condition

$$\begin{cases} \partial_{0+}^{\alpha} u(x,t) = Lu(x,t), & x\in\Omega,\ 0<t\le T,\\ u(x,t)=0, & x\in\partial\Omega,\ 0\le t\le T,\\ u(x,0)=a(x), & x\in\Omega,\\ u_t(x,0)=b(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where u is an unknown function denoting the solute concentration and ${}_0\partial_t^{\alpha} u$ is the Caputo left-sided fractional derivative of order $\alpha\in(1,2)$ defined by

$$\partial_{0+}^{\alpha} u = \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\partial_{ss}u(x,s)}{(t-s)^{\alpha-1}}ds, \qquad 1<\alpha<2,$$

in which $\Gamma(\cdot)$ is the Gamma function, and $-L$ is a symmetric uniformly elliptic operator defined on $D(-L)=H^2(\Omega)\cap H_0^1(\Omega)$ given by

$$Lu(x)=\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{d} a_{ij}(x)\frac{\partial}{\partial x_j}u(x)\Big)+c(x)u(x), \quad x\in\Omega,$$

in which the coefficients satisfy

$$a_{ij}=a_{ji},\ \ 1\le i,j\le d,\ \ a_{ij}\in C^{\infty}(\overline{\Omega}),$$

$$v\sum_{i=1}^{d}\xi_i^2\le\sum_{i,j=1}^{d}a_{ij}(x)\xi_i\xi_j,\ \ x\in\overline{\Omega},\ \ \xi\in\mathbb{R}^d,\ \ v>0,$$

$$c(x)\le 0,\ \ x\in\overline{\Omega},\ \ c(x)\in C^{\infty}(\overline{\Omega}).$$

Denote $g(x)=u(x,T)$ as the final time data. Since the measurement is noise-contaminated inevitably, we denote the noisy measurement of g as g^{δ} which satisfies

$$\|g^{\delta}-g\|\le\delta, \tag{1.2}$$

where $\|\cdot\|$ is the $L^2(\Omega)$ norm throughout this paper.

In this paper, we consider the following two backward problems:

(BP1): Suppose $b(x)$ is known, we reconstruct $a(x)$ from the measurement $g^{\delta}(x)$.

(BP2): Suppose $a(x)$ is known, we reconstruct $b(x)$ from the measurement $g^{\delta}(x)$.

To our knowledge, there is no work on the backward problem for time-fractional diffusion wave equations. The backward problem with order $\alpha\in(1,2)$ is more complicated than the one with order $\alpha\in(0,1)$ since there are two initial functions to be investigated. In this paper, we use the Tikhonov regularization method to solve the backward problems (BP1) and (BP2), respectively. Since the uniqueness for (BP1) and (BP2) may not hold, we have to consider the best-approximate solutions. In this paper, we consider originally the existence, uniqueness and conditional stability for each backward problem and present the convergence rate of the Tikhonov regularized solution approach to the best-approximate solution under an a priori assumption by using an a priori and an a posteriori regularization parameter choice rule, respectively. Numerical results for five examples in one-dimensional case and two-dimensional case are provided to verify the effectiveness of the proposed method.

This paper is organized as follows. In Section 2, we give some preliminary results. The integral equations, the ill-posedness of the backward problems (BP1) and (BP2), and the conditional stabilities are deduced in Section 3. In Section 4, we present the Tikhonov regularization method and give the convergence rate for each regularized solution. Numerical examples are given in Section 5. A brief conclusion is given in Section 6.

2. Preliminaries

Throughout this paper, we use the following definitions and lemmas.

Definition 2.1 (*[31,32]*). The generalized Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ \ z\in\mathbb{C},$$

where $\alpha>0$, $\beta\in\mathbb{R}$.

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