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Long time stability and convergence rate of MacCormack rapid solver method for nonstationary Stokes–Darcy problem

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ABSTRACT

We propose and study a combination of two second-order implicit–explicit (IMEX) methods for the coupled Stokes–Darcy system that governs flows in karst aquifers. The first is a second-order explicit two-step MacCormack scheme and the second is a second-order implicit Crank–Nicolson method. Both algorithms only require the solution of two decoupled problems at each time step, one Stokes and the other Darcy. This combination so called the MacCormack rapid solver method is very efficient (faster, at least of second order accuracy in time and space) and can be easily implemented using legacy codes. Under time step limitation of the form $\Delta t \leq Ch$ (where h, Δt are mesh size and time step, respectively, and C is a physical parameter) we prove both long time stability and the rate of convergence of the method. Some numerical experiments are presented and discussed.

1. Introduction

The computational fluid dynamics (CFD) "frontier" has advanced from the simple to the complex. Generally, the simple methods taxed the available computational power when they occupied the frontier. The evaluation proceeded from methods for various forms of the potential and Navier–Stokes equations, or Stokes equations in the surface region to the Darcy's law in the subsurface region and then to nonstationary mixed Stokes–Darcy model (for example, see [1], chapters 6–8, and [2,3]) which is the subject of this work. Most of the schemes were developed at a time when the use of the Navier–Stokes equations was prohibitive for many problems because of the large computer memory or CPU time required. If the partitioned methods for such evolutionary problems were considered economical of computer resources when they were introduced, they are still so [4,5]. In this work, we consider the coupled fluid flow and porous media flow modeled by a mixed Stokes–Darcy problem. As literature on the mathematical analysis, numerical methods, and applications for the evolutionary groundwater–surface water flows, see for example [6,2,3,7,4,5,8] and the references therein.

To specify the problem considered, let Ω_f be a fluid flow domain coupled with a porous media flow in Ω_p and lie across an interface Γ from each other, where $\Omega_{f/p} \subset \mathbb{R}^d$ ($d = 2$ or 3) are bounded domains, that is, $\Omega_f \cap \Omega_p = \emptyset$ and $\Gamma = \overline{\Omega}_f \cap \overline{\Omega}_p$. Let $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$, n_f and n_p be the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$, respectively, and τ_j, $j = 1, \ldots, d-1$, be the unit tangential vectors on the interface Γ. It is worth noticing to recall that $n_p = -n_f$ on Γ.

Let T be a positive parameter (T can be equal ∞). The fluid velocity and the porous media flow are governed by the Stokes equations and the equations given in [9], respectively, that is

$$\frac{\partial u}{\partial t} - \nu\triangle u + \nabla p = f, \quad \text{in } \Omega_f \times [0,\ T], \tag{1}$$

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$$\nabla \cdot u = 0, \quad \text{in } \Omega_f \times [0, T], \tag{2}$$

$$S_0 \frac{\partial \phi}{\partial t} + \nabla \cdot q = \tilde{g}, \quad \text{in } \Omega_p \times [0, T], \tag{3}$$

$$q = -K\nabla\phi, \quad \text{in } \Omega_p \times [0, T] \ \text{(Darcy's law)}, \tag{4}$$

$$u_p = \frac{q}{\eta}, \quad \text{in } \Omega_p \times [0, T], \tag{5}$$

$$u(x, 0) = u^0(x), \ \text{in } \Omega_f \text{ and } \phi(x, 0) = \phi^0(x), \ \text{in } \Omega_p, \tag{6}$$

$$u(x, t) = \hat{u}(x, t), \ \text{in } (\partial\Omega_f \setminus \Gamma) \times [0, T] \text{ and } \phi(x, t) = \hat{\phi}(x, t), \ \text{in } (\partial\Omega_p \setminus \Gamma) \times [0, T], \tag{7}$$
$$+ \ \text{coupling conditions across } \Gamma,$$

where $u(x, t)$, $p(x, t)$, $f(x, t)$ and ν represent the velocity of the fluid flow in Ω_f, the kinetic pressure, the external force and the kinematic viscosity, respectively. Similarly: $\phi(x, t)$, $u_p(x, t)$, $\tilde{g}(x, t)$, K, S_0, η, and $q(x, t)$ are the piezometric head, the fluid velocity in Ω_p, the source term, the hydraulic conductivity tensor, the specific mass storativity, the volumetric porosity and the specific discharge defined as the volume of the fluid flowing per unit time through a unit cross-sectional area normal to the direction of the flow, respectively. In addition, we denote by $\phi = \bar{z} + \frac{p_p}{\rho g}$, the sum of elevation head plus pressure head, where p_p is the pressure of the fluid in Ω_p, ρ is the fluid density, g is the gravitational acceleration and $\bar{z}$ is the elevation from a reference level. Since the formulae can become quite heavy, for the sake of simplicity, we assume in the following that $\bar{z} = 0$. Furthermore, we assume that $K = diag(k, k, \ldots, k)$ with $k \in L^\infty(\Omega_p)$, $k > 0$, which implies that the porous media is isotopic. Finally, using the Darcy's law (4) in the continuity equation (3) in Ω_p yields the following partial differential equation

$$S_0 \frac{\partial \phi}{\partial t} - \nabla \cdot (K\nabla\phi) = \tilde{g}, \quad \text{in } \Omega_p \times [0, T].$$

The boundary conditions given by (7) are not very important to either the analysis or algorithm studied herein. For simplicity, we assume in this work the homogeneous Dirichlet boundary conditions for the coupled model, that is

$$\hat{u} = 0, \ \text{in } (\partial\Omega_f \setminus \Gamma) \times [0, T] \ \text{ and } \ \hat{\phi} = 0, \ \text{in } (\partial\Omega_p \setminus \Gamma) \times [0, T].$$

Since $n_{f/p}$ denote the indicated outward pointing unit normal vectors on Γ, the coupling conditions are conservation of mass given by

$$u \cdot n_f + u_p \cdot n_p = 0 \Leftrightarrow u \cdot n_f - \frac{1}{\eta} K\nabla\phi \cdot n_p = 0, \ \text{on } \Gamma, \tag{8}$$

and balance forces

$$p - \nu n_f \cdot \nabla u \cdot n_f = \rho g \phi, \ \text{on } \Gamma. \tag{9}$$

The equivalence given by (8) comes from the Darcy's law (4) and relation (5). The last condition needed is a tangential condition on the fluid region's velocity on the interface:

$$-\nu\tau_j \frac{\partial u}{\partial n_f} = \frac{\alpha}{\sqrt{\tau_j \cdot K \cdot \tau_j}} (u - u_p) \cdot \tau_j, \ \ j = 1, \ldots, d-1, \ \text{on } \Gamma. \tag{10}$$

There have been many discussions on condition (10). However, it has been observed that in practice the term $u_p \cdot \tau_j$, on the right hand side of relation (10) is much smaller than other terms and is thus negligible. This leads to the Beavers–Joseph–Saffman interfacial coupling [10,7,11]

$$-\nu\tau_j \frac{\partial u}{\partial n_f} = \frac{\alpha}{\sqrt{\tau_j \cdot K \cdot \tau_j}} u \cdot \tau_j, \ \ j = 1, \ldots, d-1, \ \text{on } \Gamma.$$

In this work, we focus on the numerical solution of the mixed model (1)–(7) using MacCormack rapid solver (MCRS) method, that is, a coupled Crank–Nicolson scheme and explicit MacCormack method. There are many reasons as discussed in [12] that have led to active research on developing effective and efficient decoupling techniques for multiphysics mixed models so that existing single model solvers can be applied locally with little extra computational and software overhead. In the literature (see for example [13,3,12]) most of the decoupled methods for the Stokes–Darcy model are developed for the stationary case. For the nonstationary case, parallel non-iterative multi-physics domain decomposition methods are proposed and numerical experiments are reported in [14,15]. Here, the MCRS method is proposed for devising decoupled marching algorithms for the mixed model so that at each time level, two decoupled subproblems are solved independently by invoking a Stokes solver and a Darcy solver, respectively. However, the method we study is new, three step partitioned scheme motivated by the form of the coupling. It involves second order implicit discretization of the subdomain terms and two step explicit MacCormack discretization of the exactly skew symmetric coupling terms. Furthermore, it is a satisfactory

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