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Exponential integrator methods for systems of non-linear space-fractional models with super-diffusion processes in pattern formation

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ABSTRACT

Nonlocality and spatial heterogeneity of many practical systems have made fractional differential equations very useful tools in Science and Engineering. However, solving these type of models is computationally demanding. In this paper, we propose an exponential integrator method for space fractional models as an attractive and easy-to-code alternative for other existing second-order exponential integrator methods. This scheme is based on using a real distinct poles discretization for the underlying matrix exponentials. One of the major benefits of the proposed scheme is that the algorithm could be easily implemented in parallel to take advantage of multiple processors for increased computational efficiency. The scheme is established to be second-order convergent; and proven to be robust for nonlinear space fractional reaction–diffusion problems involving non-smooth initial data. Our approach is exhibited by solving a system of two-dimensional problems which exhibits pattern formation and has applications in cell-division. Empirically, super-diffusion processes are displayed by investigating the effect of the fractional power of the underlying Laplacian operator on the pattern formation found in these models. Furthermore, the superiority of our method over competing second order ETD schemes, BDF2 scheme, and IMEX schemes is demonstrated.

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1. Introduction and preliminaries

Many complex systems are modeled by fractional order derivatives. These non-integer derivatives have become popular in recent times due to the fact that they provide an adequate description of many processes that display anomalous diffusion. Various applications are in modeling of different phenomena such as nanotechnology, control theory of dynamical systems, viscoelasticity, anomalous transport and anomalous diffusion, financial modeling, random walk, and biological modeling, see [1–5]. More detailed work on physical and engineering processes with applications of fractional calculus can be found in [4,6–8]. Furthermore, sub-diffusion (fractional in time) and super-diffusion (fractional in space) have been observed and the effect of the fractional orders have been seen in the solution profiles in many models, see [9,10].

Despite many advantages of using fractional derivatives, numerical solutions are computationally demanding. Existence and uniqueness of solutions to fractional reaction–diffusion equations have been investigated using some assumptions on reaction term (such as local/global Lipschitz continuity) in [11,12]. Researchers have proposed many numerical approaches to solve equations involving fractional order derivatives. Among such methods are finite difference, finite element or finite

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volume discretization of the fractional operator, combined with a semi-implicit Euler formulation for the time evolution of the solution. Specifically, for space fractional equations many approaches such as Krylov methods, fast numerical integration in conjunction with effective preconditioners and Fourier spectral methods have been introduced in [9,13,14], see also [11,12,15].

According to Samko et al. in [16], a fractional power of the Laplace operator is defined as follows:

$$-(-\Delta)^{\alpha/2}w(x) = -\mathcal{F}^{-1}|x|^{\alpha}\mathcal{F}w(x), \tag{1.1}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse, respectively.

Definition 1.1. The Riesz fractional derivative of function w with order $m-1<\alpha\le m$, $m\ge 1$ is defined as [17]

$$\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}w(x) = -\frac{c_{\alpha}}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_{-\infty}^{+\infty}|x-\xi|^{m-\alpha-1}w(\xi)d\xi, \tag{1.2}$$

where

$$c_{\alpha} = \frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)}, \qquad \alpha\neq 1.$$

Lemma 1.2 (*[18]*). *For a function $w(x)$ defined on the infinite domain $-\infty<x<\infty$, the following equality holds:*

$$-(-\Delta)^{\alpha/2}w(x) = -c_{\alpha}\left[{}_{-\infty}D_x^{\alpha}w(x)+{}_xD_{+\infty}^{\alpha}w(x)\right] = \frac{\partial^{\alpha}}{\partial|x|^{\alpha}}w(x). \tag{1.3}$$

Here, ${}_{-\infty}D_x^{\alpha}w(x)$ and ${}_xD_{+\infty}^{\alpha}w(x)$ are the left-sided and right-sided the Riemann–Liouville fractional derivatives given as:

$${}_{-\infty}D_x^{\alpha}w(x) = \frac{1}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial x^m}\int_{-\infty}^{x}\frac{w(\xi)}{(x-\xi)^{\alpha+1-m}}d\xi, \tag{1.4}$$

and

$${}_xD_{+\infty}^{\alpha}w(x) = \frac{1}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial x^m}\int_{x}^{+\infty}\frac{w(\xi)}{(\xi-x)^{\alpha+1-m}}d\xi. \tag{1.5}$$

In this paper, we consider systems of nonlinear Riesz space fractional reaction–diffusion equations with homogeneous Dirichlet boundary conditions given as

$$\frac{\partial u}{\partial t} + K_1\frac{\partial^{\alpha}u}{\partial|x|^{\alpha}} + K_2\frac{\partial^{\alpha}u}{\partial|y|^{\alpha}} = g_1(u,v),\ (x,y,t)\in\Omega\times(0,T] \tag{1.6}$$

$$\frac{\partial v}{\partial t} + K_3\frac{\partial^{\alpha}v}{\partial|x|^{\alpha}} + K_4\frac{\partial^{\alpha}v}{\partial|y|^{\alpha}} = g_2(u,v),\ (x,y,t)\in\Omega\times(0,T], \tag{1.7}$$

with initial conditions

$$u(x,y,0) = u_0(x,y),\ (x,y)\in\Omega\cup\partial\Omega, \tag{1.8}$$

$$v(x,y,0) = v_0(x,y),\ (x,y)\in\Omega\cup\partial\Omega \tag{1.9}$$

where K_1, K_2, K_3, and K_3 are diffusion coefficients, Ω is bounded in $\mathbb{R}^2$, $1<\alpha\le 2$ and $\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}$ represents the Riesz fractional derivative. We assume that g_1 and g_2 are sufficiently smooth functions to ensure that the problem with the specified initial and boundary conditions possesses unique solutions [11].

We propose an exponential integrator method for space fractional models as an attractive and easy-to-code alternative for other existing second-order methods. This scheme is based on using a real, distinct poles discretization for the underlying matrix exponentials. A benefit of the proposed scheme is that the algorithm could be easily implemented in parallel for increased computational efficiency. The scheme is established to be second-order convergent; and proven to be robust for system of nonlinear space fractional reaction–diffusion problems involving non-smooth initial data. Our approach is exhibited by solving system of two-dimensional problems which exhibit pattern formation and have applications in cell-division. These include the system of space-fractional Schnakenberg model and the system of space-fractional Gray–Scott model. Empirically, super-diffusion processes are displayed by investigating the effect of the fractional power of the underlying Laplacian operator on the pattern formation found in these models. Furthermore, the superiority of our method over competing second order ETD schemes, BDF2 scheme, and IMEX schemes is demonstrated. Our experiments confirm that the proposed scheme is computationally more efficient.

2. Spatial discretization methods for fractional derivative

In this section, a fractional centered differencing and the matrix transfer technique are introduced for the discretization of the Riesz space-fractional derivative.

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