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A novel approach of high accuracy analysis of anisotropic bilinear finite element for time-fractional diffusion equations with variable coefficient

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ABSTRACT

In this paper, by using bilinear finite element and $L1$ approximation, a fully-discrete scheme is established for time fractional diffusion equation with variable coefficient on anisotropic meshes. Unconditionally stable analysis of the proposed scheme are presented in both L^2-norm and H^1-norm. Moreover, convergence, superclose and superconvergence results are derived by combining interpolation with projection, which is the key technique for the numerical analysis. Specifically, by defining a novel projection operator, the error estimate between the projection and the exact solution is obtained on anisotropic meshes. Furthermore, high accuracy analysis on interpolation of bilinear finite element and projection is gained by means of some known results about the interpolation and mean value technique. Based on the related results about projection and interpolation, the optimal error estimate in L^2-norm and superclose of interpolation in H^1-norm are deduced by skillfully dealing with fractional derivative. At the same time, the global superconvergence is presented by employing interpolation postprocessing operator. Finally, numerical results are provided to demonstrate the validity of the theoretical analysis.

1. Introduction

In recent years, many fractional partial differential equations (FPDEs) are used to describe some phenomena or processes with memory and hereditary in different fields, such as ecology, biology, chemistry and biochemistry, finance, complex network, the nonlinear oscillation of earthquake, electrode-electrolyte polarization, and so on (see [1–6]).

In this paper, we consider the following two-dimensional (2D) time-fractional diffusion equations with variable coefficient, which are derived from the standard diffusion equations by replacing the first order derivative of time with a fractional derivative of order α:

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} - \nabla\cdot(\omega(X)\nabla u) = f(X,t), & (X,t)\in \Omega\times(0,T], \\ u(X,t)=0, & (X,t)\in\partial\Omega\times(0,T], \\ u(X,0)=u_0(X), & X\in\Omega, \end{cases} \tag{1}$$

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where $\Omega \subset R^2$ is a bounded convex polygonal region with boundary $\partial\Omega$, $X = (x, y)$, there exist positive constants ω_1, ω_2 such that $0 < \omega_1 \le \omega(X) \le \omega_2$, $u_0(X)$ and $f(X,t)$ are given functions assumed to be sufficiently regular and $\frac{\partial^\alpha u}{\partial t^\alpha}$ is Caputo fractional derivative defined by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(X,s)}{\partial s} \frac{ds}{(t-s)^\alpha},\ 0 < \alpha < 1.$$

The time-fractional diffusion equation is a useful tool for modeling a wide range of important physical phenomena. As is known, it is not easy to obtain analytic solutions. Then, developing reliable numerical methods for TFDEs becomes more and more necessary and important. For example, finite difference methods (see [7–12]), spectral methods (see [13–16]), meshless methods (see [17,18]), Petrov–Galerkin method (see [19]), DG methods (see [20]), finite element methods (see [21,22]), and so on.

Finite difference methods are commonly used to solve TFDEs. For example, Lv gave a high order finite difference method for 1D TFDEs in [7]. Diego developed an implicit unconditionally stable finite difference scheme in [8]. Li proposed an implicit finite difference scheme for solving the forward problem in [9]. Cui proposed a compact ADI scheme and discussed its stability and accuracy for 2D TFDEs in [10]. [11] presented the stability and convergence analysis based on a maximum principle for a finite difference scheme. A fully-discrete difference scheme was constructed with space discretization by compact difference method. Error estimates were established for two kinds of nonuniform meshes in [12].

As for spectral methods for 1D TFDEs, there are some literatures. For example, Legendre polynomials were adopted in temporal discretization and the Fourier-like basis functions were constructed for spatial discretization in [13]. [14] established a finite difference scheme in time and Legendre spectral methods in space, moreover stability and convergence of the method were rigorously discussed. [15] presented a spectral tau algorithm based on Jacobi operational matrix, and five numerical examples demonstrated the validity and effectiveness of the method. [16] researched a pseudo-spectral scheme and presented semi-discrete system of TFDEs in an accurate and stable way. Meshless method is an efficient numerical tool for fractional partial differential equations. [17] developed a novel boundary meshless approach and Laplace transformed boundary particle method. The discrete system of FPDEs was obtained by using the meshless shape functions and the meshless collocation formulation, and the stability and convergence of this meshless approach were investigated theoretically and numerically in [18]. Galerkin method is a classical numerical method for PDEs. And, various forms of Galerkin methods have been developed to fractional PDEs. For example, a time-stepping discontinuous Petrov–Galerkin method was proposed and analyzed in [19], which combined with the continuous conforming finite element method in space. An implicit fully-discrete direct discontinuous Galerkin finite element method was considered for solving TFDEs in [20]. Galerkin finite element approximation of optimal control problems governed by TFDEs was investigated in [21]. [22] analyzed a space semi-discrete scheme based on the standard Galerkin finite element method using continuous piecewise linear functions and derived optimal error estimates for both cases of initial data and inhomogeneous term.

All of the above literatures focus on TFDEs with constant coefficients. However, there are limited studies on FDEs with variable coefficients. For example, Cui established high order compact exponential finite difference scheme for solving the time fractional convection–diffusion–reaction equation with variable coefficients [23,24], respectively. Difference schemes of the second and fourth order of approximation in space and the second order in time for TFDEs with variable coefficients are constructed in [25]. [26] considered the numerical approximation of TFDEs with variable coefficients on a semi-infinite spatial domain. A fully-discrete scheme based on finite difference method in time and spectral approximation using Legendre functions in space is proposed. [27] set up time-stepping discontinuous Galerkin method to numerically solve TFDEs with variable coefficients and derived well-posed of the fully-discrete scheme and error result. Unfortunately, the above literatures on FDEs with variable coefficients only discussed convergence analysis about space step, which paid main attention to finite difference method.

As is known to all, superconvergence is an efficient procedure for improving the accuracy of approximation of many problems in numerical analysis of finite element methods. For single-term time FPDEs, there are a few works on superconvergence analysis. For example, by using projection of [28]. [29,30] studied the hybridizable discontinuous Galerkin (HDG) method for the spatial discretization of time fractional diffusion models and obtained a nodal superconvergence result. By use of the bilinear interpolation alone, [31,32] gave a global superconvergence theory for FEMs by use of the properties of integral identities. We have finished some work about the global superconvergence of integer order partial differential equations (see [33–37]). Recently, [38,39] derived global superconvergence results of nonconforming FEMs and mixed FEMs for TFDEs with constant coefficients, respectively. By the way, almost all of the above investigations are based on the regularity assumption or quasi-uniform assumption [40] on the meshes, i.e., there exists a positive constant C which is independent of h and satisfies $h_K/\rho_K \le C$ or $h/\tilde{h} \le C$, where h_K and ρ_K denote the diameter and the radius of inscribed circle of the element K, respectively, $h = \max_K h_K$, $\tilde{h} = \min_K h_K$. However, the solution of the problem considered may have anisotropic behavior in some parts of the domain when the boundary or the interior layers appear, i.e., the solution may vary significantly in certain direction. To reflect this anisotropy, it is a better choice to use anisotropic meshes with a small mesh size in the rapid variation direction of the solution and a larger mesh size in the perpendicular direction. However, it seems that there are few studies focusing on the anisotropic FEM for problem (1).

In the present work, based on interpolation combination projection, we derive the superclose and superconvergence properties of the bilinear FE for the time fractional diffusion equation with variable coefficients (1) under anisotropic meshes. Firstly, on the one hand, with help of the element's anisotropic interpolation property, the error estimate about the new