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Godunov-type upwind flux schemes of the two-dimensional finite volume discrete Boltzmann method

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ABSTRACT

A simple unified Godunov-type upwind approach that does not need Riemann solvers for the flux calculation is developed for the finite volume discrete Boltzmann method (FVDBM) on an unstructured cell-centered triangular mesh. With piecewise-constant (PC), piecewise-linear (PL) and piecewise-parabolic (PP) reconstructions, three Godunov-type upwind flux schemes with different orders of accuracy are subsequently derived. After developing both a semi-implicit time marching scheme tailored for the developed flux schemes, and a versatile boundary treatment that is compatible with all of the flux schemes presented in this paper, numerical tests are conducted on spatial accuracy for several single-phase flow problems. Four major conclusions can be made. First, the Godunov-type schemes display higher spatial accuracy than the non-Godunov ones as the result of a more advanced treatment of the advection. Second, the PL and PP schemes are much more accurate than the PC scheme for velocity solutions. Third, there exists a threshold spatial resolution below which the PL scheme is better than the PP scheme and above which the PP scheme becomes more accurate. Fourth, besides increasing spatial resolution, increasing temporal resolution can also improve the accuracy in space for the PL and PP schemes.

1. Introduction

The conventional lattice Boltzmann method (LBM) solves the lattice Boltzmann equation (LBE) in a Lagrangian space by coupling the discretization of the particle velocity space and configuration space. With such a coupling mechanism, the Courant–Friedrichs–Lewy (CFL) number can be chosen to be exactly one globally, which means that after each streaming step, the particle distribution functions (PDFs) along all lattice velocities will stop perfectly at grid points. Such a unique feature allows the LBM to achieve second-order accuracy in space with a first-order advection scheme. However, it is this same feature that makes the LBM suffer from several pitfalls, one of which is that the LBM cannot perfectly capture curved boundaries due to its uniform mesh structure [1]. Some pioneering work [2–5] showed that the LBE can simply be considered a special finite-difference version of the more generalized discrete Boltzmann equation (DBE) that is Eulerian in nature. Therefore, one can completely avoid the velocity–configuration coupling by solving the DBE instead of the LBE, which subsequently enables the use of an arbitrary mesh. Following this idea, many Eulerian discrete Boltzmann methods (DBM) have been developed to incorporate complex geometries. Among these, the finite volume discrete Boltzmann method (FVDBM) [6–22] has received the most attention due to the built-in conservative property of the finite volume method (FVM).

Nevertheless, a considerable diffusion error has been, expectedly, observed when the DBE is solved, and especially on irregular grids [6,23]. Such an error exclusively comes from the evaluation of the advection in Eulerian space. First, the

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CFL = 1 condition will be lost when the DBE is solved in Eulerian space, which means numerical viscosity has to be introduced by upwind schemes (or a combination of the upwind scheme and others) to maintain stability [24]. The second reason, which is not unique to the DBE, is that the complex topology on an unstructured mesh will inevitably introduce some error to the advection stencil, part of which eventually becomes numerical viscosity. These explain why, in a previous study, a theoretically second-order scheme for advection could only deliver a result that is slightly higher than first order [25]. As a result, the DBE approaches have an edge over LBE approaches, which is the ability to handle complex geometries with unstructured mesh. However, the consequence for this advantage is a significant increase in numerical error. Therefore, there is a need for higher-order advection schemes in order to use the DBM as a better alternative to the LBM [26].

The DBE is a hyperbolic equation with a strong advection term. Solving a hyperbolic equation, e.g. the Euler equation or Navier–Stokes (NS) equation, with minimum diffusion error while maintaining stability is a constant concern in the computational fluid dynamics (CFD) community. After decades of development, there are many successful techniques, among which Godunov's method has dominated many CFD codes in the subgenre of FVM, due to its higher fidelity and better stability. In Godunov's method, the advected scalar is considered as a wave moving at its characteristic velocity. Then, a Riemann problem appears at the interface between two adjacent cells, which is solved by exact or approximate Riemann solvers, e.g. Roe's solver [27]. Different reconstructions of the wave structure determine the order of Godunov's method. The piecewise-constant (PC) reconstruction proposed in the original work from Godunov [28], the piecewise-linear (PL) method developed by Bram van Leer [29–33] that gave birth to the still popular Monotone Upstream Scheme for Conservation Laws (MUSCL), and the piecewise-parabolic (PP) reconstruction introduced by Colella and Woodward [34,35] give the first-order, second-order and third-order Godunov's methods, respectively.

Despite the high success of Godunov's method in the CFD community, its importance is not recognized widely within the LBM circle. Most FVDBM [6–18,22] work treats the advected scalar in the advection term as a scalar value in a static point of view, in contrast to Godunov's method. The only application of Godunov's method in FVDBM so far was employed by Patil and Lakshmisha in their simulations of single-phase problems [19–21], which involve a Riemann solver and a limiter that satisfies the Total Variation Diminishing (TVD) property. However, the linear advection term in the DBE (i.e. the advection has a constant speed that is defined by the lattice velocity) and the mutual independence among all PDFs do not require any type of Riemann solver when calculating the PDF flux on the face between two neighbor cells (this will be explained in detail later in this paper). The TVD limiters, which were originally developed for simulating one-dimensional (1D) shocks in CFD tools, were first introduced into LBM simulations by Teng et al. [36] and Lee et al. [37] to solve the streaming step of LBE for multi-phase problems that experience sharp gradients similar to acoustic shocks. Therefore, TVD limiters are not necessary when the flows are single-phase and do not have large gradients if simulated by the DBE. More importantly, it was pointed out very early in the CFD community by Goodman and LeVeque that TVD limiters are no better than first-order accurate when extended to multiple dimensions [38]. This is probably why only an overall first-order accuracy in space was reported for their flux scheme with TVD limiters in the work of Patil and Lakshmisha [19–21].

Therefore, in order to achieve better accuracy in space, a simple Godunov-type upwind approach that does not require any Riemann solver for the advection of FVDBM is developed in the present paper. Then, with different PDF wave reconstructions (PC, PL and PP) on a universal stencil, three Godunov flux schemes with different orders of accuracy are formulated. After that, a semi-implicit temporal scheme specifically designed for the presented Godunov-type flux scheme is shown. In order to make comparisons between the Godunov and non-Godunov schemes, a standard second-order upwind (SOU) scheme, which is non-Godunov, and the corresponding time-marching approach are also provided. Next, a boundary treatment that works seamlessly with all of the developed flux schemes is also established. With thorough numerical testing, some important conclusions can be reached.

2. Formulation of the FVDBM

The DBE with the Bhatnagar–Gross–Krook (BGK) collision model, which is obtained by discretizing the particle velocity space of the continuous Boltzmann equation with a finite number of velocity components [39], is shown as follows:

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{e}_\alpha \cdot \nabla f_\alpha = -\frac{1}{\tau}\left(f_\alpha - f_\alpha^{eq}\right) \quad \alpha = 0, 1, 2, \ldots, M-1 \tag{1}$$

where f and f^{eq} are the PDF and equilibrium PDF respectively, and $\boldsymbol{e}$ is the lattice velocity. The subscript α indicates the αth member out of M total components of the discretized particle velocity space, and τ is the relaxation time. By choosing a proper lattice model, $\boldsymbol{e}_\alpha$ and f_α^{eq} can be defined explicitly. For example, for the commonly used D2Q9 lattice, they are defined as:

$$\boldsymbol{e}_\alpha = \begin{cases} [0, 0]c & \alpha = 0 \\ \left[cos\left((\alpha-1)\frac{\pi}{2}\right), sin\left((\alpha-1)\frac{\pi}{2}\right)\right]c & \alpha = 1-4 \\ \left[cos\left((\alpha-5)\frac{\pi}{2}+\frac{\pi}{4}\right), sin\left((\alpha-5)\frac{\pi}{2}+\frac{\pi}{4}\right)\right]\sqrt{2}c & \alpha = 5-8 \end{cases} \tag{2}$$

$$f_\alpha^{eq} = \omega_\alpha \rho \left[1 + \frac{\boldsymbol{e}_\alpha \cdot \boldsymbol{u}}{c_s^2} + \frac{(\boldsymbol{e}_\alpha \cdot \boldsymbol{u})^2}{2c_s^4} - \frac{\boldsymbol{u} \cdot \boldsymbol{u}}{2c_s^2}\right] \tag{3}$$

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