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# An overdetermined problem for a weighted Poisson's equation[☆]

Qi-hua Ruan [a], Weihua Wang [a,b,*], Qin Huang [a]

[a] School of Mathematics and Finance, Putian University, Putian 351100, PR China
[b] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100049, PR China

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ABSTRACT

In this paper, we discuss an overdetermined problem for a weighted Poisson's equation. We prove that if there exists a solution of the weighted Poisson's equation $\triangle u + \nabla \log w \cdot \nabla u = -1$ on a smooth bounded domain $\Omega$ with both Dirichlet and Neumann constant boundary condition, and the weight function $w$ satisfies some conditions in $\Omega$, then $\Omega$ is a ball. We also study some applications of the overdetermined problems and some overdetermined problems with nonconstant Neumann boundary condition.



## 1. Introduction and main results

Let $\Omega$ be a smooth connected and bounded domain in $\mathbb{R}^n$ and consider the Poisson's equation $\Delta u = -1$ in $\Omega$ with Dirichlet condition $u = 0$ or Neumann condition $u_\nu = c$ on $\partial\Omega$; here $\nu$ denotes the unit outer normal vector of $\partial\Omega$ and $c$ is a constant. It is well known there exists a solution of the Poisson's equation with Dirichlet condition or Neumann condition. But both Dirichlet condition and Neumann condition cannot be imposed simultaneously. In general, the resulting problem does not admit a solution unless $\Omega$ is a ball. Indeed in a celebrated paper [1], Serrin proved that there exists a solution of the following overdetermined problem:

$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0 \text{ and } u_\nu = c & \text{on } \partial\Omega \end{cases}$$

if and only if $\Omega$ is a ball and $u$ is a radial function.

This result has its physical explanation [1,2], the associated Dirichlet problem describes a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross sectional form $\Omega$. If we fix rectangular coordinates $(x, y, z)$ with $z$-axis directed along the pipe, it is well known that the flow velocity $u$ along the pipe is then a function of $x$ and $y$, and satisfies $\Delta u + k = 0$, where $k$ is a constant related to the viscosity and density of the fluid. The adherence condition is given by $u = 0$ on $\partial\Omega$. The result of Serrin allows us to state that the tangent stress per unit area on the pipe wall (represented by $\mu u_\nu$, where $\mu$ is the viscosity) is the same at all points of the wall if and only if the pipe has a circular cross section.

The main tool of Serrin's proof is a technique known as the moving planes method, combined with a refinement of the maximum principle. Later Weinberger [3] used the maximum principle applied to an auxiliary function to obtain a

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* Corresponding author at: School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100049, PR China.
*E-mail addresses:* ruanqihua@163.com (Q.-h. Ruan), wangvh@163.com (W. Wang), qinhuang78@163.com (Q. Huang).

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very simple proof of the same result. There are some alternative proofs (see [4–8]). Further many extensions have been studied, concerning domains with a less regular boundary (see [9,10]), or exterior domain (see [11–14]), or annular domains (see [15,16]). Other extensions have been given for elliptic equations more general than the Laplacian and possibly degenerated [17–23] and for parabolic equations (see [24]). In [16,25–27], the authors consider the overdetermined problems where the Neumann boundary condition is not constant and involves the boundary curvature. In [6,28], the overdetermined problems in the constant curvature space are discussed. Very recently, Bianchini and Ciraolo [29] considered Serrin problem with the anisotropic p-Laplacian operator in the same spirit as [4] and [18]. The literature about overdetermined problems is so wide that it is impossible to report it exhaustively.

In this paper, we investigate the overdetermined problem

$$\begin{cases} \triangle_w u = -1 & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 \text{ and } u_\nu = c & \text{on } \partial\Omega, \\ \nabla u(0) = 0, & \\ \nabla^2 \log w(\nabla u, \nabla u) + \dfrac{(\nabla \log w \cdot \nabla u)^2}{\alpha} \le 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where the weighted Laplacian is defined by $\triangle_w = \triangle + \nabla \log w \cdot \nabla$, for a positive function $w \in C^{2,\gamma}(\mathbb{R}^n \setminus \{0\})$ that is a homogeneous function of degree $\alpha > 0$, i.e. $\nabla w \cdot x = \alpha w$, (for example, $w = |x|^\alpha$).

**Remark 1.1.** The condition of $\nabla u(0) = 0$ is not essential. Since the weighted Poisson equation of $\triangle_w u = -1$ can be understood by weak solution,i.e. it satisfies that for any $\varphi \in C_0^\infty(\Omega)$, $\int_\Omega \nabla u \cdot \nabla\varphi d\mu = \int_\Omega \varphi d\mu$, where $d\mu = w dx$, and pointwise equations in all subsequent calculations can be made in the sense of integration. Hence the only singular point of $x = 0$ may be neglected.

Notice that $\triangle_w u = \frac{1}{w} div(w\nabla u)$. To our knowledge, there are few references concerning this kind of overdetermined problems. One difficulty of this problem consists in finding suitable weight functions such that a solution exists. As we known, if $w$ is $\alpha$-homogeneous and we take a radial solution of $\triangle u = -A$ in $B(0, R)$ such that $u = 0$ on $\partial B(0, R)$, then $\triangle u$ and $\nabla \log w \cdot \nabla u$ must be both constant and $u$ satisfies problem (1.1) for a suitable choice of the constant $A$. This fact implies us to choose a weight function $w$ to be a homogeneous function. Our choice of the weight function is also motivated by the Ref. [30], where the authors use the ABP method to get the sharp weighted isoperimetric inequalities. In [31], the first author obtain a sharp gradient estimate for the solution of the weighted Laplace's equation.

Then we obtain the following result about the overdetermined problem (1.1).

**Theorem 1.1.** *Let $\Omega$ be a connected and bounded domain with the smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. If there exists a solution $u \in C^2(\Omega)$ of the overdetermined problem (1.1), then $\Omega$ is a ball centered at 0.*

When $u$ is a radial symmetric solution, $\nabla w \cdot x = \alpha w$ implies $\nabla^2 \log w(\nabla u, \nabla u) + \frac{(\nabla \log w \cdot \nabla u)^2}{\alpha} \le 0$. In fact, by homogeneity of the weight function $w$, it is easy to see that $\nabla w$ is homogeneous of degree $\alpha - 1$, then

$$\begin{aligned} &\nabla^2 \log w(\nabla u, \nabla u) + \frac{(\nabla \log w \cdot \nabla u)^2}{\alpha} \\ =& \frac{1}{r^2}(\frac{\partial u}{\partial r})^2(\frac{1}{w}\nabla^2 w(x, x) - \frac{\alpha - 1}{\alpha}(\nabla \log w \cdot x)^2) \\ =& \frac{1}{r^2}(\frac{\partial u}{\partial r})^2(\alpha(\alpha - 1) - \alpha(\alpha - 1)) = 0. \end{aligned}$$

Thus we get the following corollary.

**Corollary 1.1.** *Let $\Omega$ be a connected and bounded domain with the smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. If there exists a radial solution $u \in C^2(\Omega)$ of the overdetermined problem (1.1), then $\Omega$ is a ball centered at 0.*

Next we give a result of the existence of a solution of the overdetermined problem (1.1).

**Theorem 1.2.** *Let $\Omega$ be a ball centered at 0 with a radius $R$ in $\mathbb{R}^n$. Then $\frac{R^2 - |x|^2}{2(n+\alpha)}$ is a solution of the overdetermined problem (1.1).*

As its application of Theorem 1.2, we deduce a mean value property for a weighted harmonic function in a ball. A function $h \in C^2(\Omega)$ is called a weighted harmonic function if it satisfies $\triangle_w h = 0$ in $\Omega$. When the weight function is a constant, the weighted harmonic function becomes a harmonic function.

The weighted harmonic function has some similar properties as the harmonic function. For example, it is well known, every bounded harmonic function in Euclidean space is a constant (see [32]). This result was extended by Yau [33] to a Riemannian manifold with a nonnegative Ricci curvature. For a weighted harmonic function, Li [34] proved this property also holds in a Riemannian manifold with a nonnegative Bakry–Emery Ricci curvature (also see [35]). The mean value property of a harmonic function is a classical result. However to our knowledge, the mean property of a weighted harmonic function is unknown so far.

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