ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Particular solutions of products of Helmholtz-type equations using the Matern function

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ARTICLE INFO

Article history:
Received 12 August 2017
Received in revised form 5 December 2017
Accepted 29 January 2018
Available online xxxx

Keywords:
Helmholtz-type equations
Radial basis function
Matern functions
Fundamental solution
Particular solution

ABSTRACT

We derive closed-form particular solutions for Helmholtz-type partial differential equations. These are derived explicitly using the Matern basis functions. The derivation of such particular solutions is further extended to the cases of products of Helmholtz-type operators in two and three dimensions. The main idea of the paper is to link the derivation of the particular solutions to the known fundamental solutions of certain differential operators. The newly derived particular solutions are used, in the context of the method of particular solutions, to solve boundary value problems governed by a certain class of products of Helmholtz-type equations. The leave-one-out cross validation (LOOCV) algorithm is employed to select an appropriate shape parameter for the Matern basis functions. Three numerical examples are given to demonstrate the effectiveness of the proposed method.

1. Introduction

The boundary element method (BEM) has been considered as one of the traditional numerical methods for solving partial differential equations. In recent years, boundary meshless methods such as the method of fundamental solutions (MFS) [1,2], singular boundary method (SBM) [3], non-singular method of fundamental solutions [4], boundary knot method (BKM) [5], etc. have become very popular for solving homogeneous partial differential equations. A common feature of these numerical methods is the need of a fundamental solution for a given differential equation. In order to extend these methods to solve inhomogeneous equations, we need to evaluate the particular solution. Coupled with the dual reciprocity method (DRM) [6] where a closed-form particular solution is required, the BEM can be extended to solve the inhomogeneous problems without domain discretization. Golberg *et al.* [7] extended the MFS to inhomogeneous problems by using the closed-form particular solution through the use of radial basis functions (RBFs). Hence, for all these types of the boundary methods, the closed-form particular solution to the inhomogeneous problems is as important as the fundamental solution to the homogeneous problems. The success of extending these boundary methods to solving inhomogeneous problems largely relies on the efficient evaluation of the particular solution. However, similar to the derivation of fundamental solutions, the derivation of the closed-form particular solution for a general differential operator is a non-trivial task. Before 1998, the closed form particular solution was restricted to the Laplace and biharmonic operators using RBFs. Chen and Rashed [8] were the first to break this barrier by constructing a closed-form particular solution of Helmholtz-type operators using Thin Plate Splines (TPS). The availability of the closed-form particular solution of Helmholtz-type equations has made significant progress for

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https://doi.org/10.1016/j.camwa.2018.01.038

effectively solving time-dependent problems [2]. A series of works has been followed up along this line using RBFs. As a result, due to the intensive research related to the closed-form particular solutions, new novel numerical techniques such as the method of particular solutions [9,10], the MFS–MPS [11], and the localized method of approximate particular solutions (LMAPS) [12] have been developed for solving general types of partial differential equations.

It is known that the accuracy of the particular solution depends on the basis function being used. Note that the particular solution requires that it satisfies the governing equation without satisfying the boundary conditions. Hence, the particular solution is not unique. A rich variety of techniques can be applied to derive the needed particular solution. Despite the intensive research in this area, it is known that the closed-form particular solutions for Helmholtz-type operators are available only for polyharmonic splines [13–15] and compactly supported radial basis functions (CS-RBFs) in 3D [16]. Even though the particular solution using polyharmonic splines works well for the MPS, it falls short for the LMPAS which is a localized method. The higher convergence rate of the RBFs such as Multiquadric (MQ) function is needed so that the obtained particular solution can be useful for the implementation for the LMAPS. We found that the Matern function has some attractive features that can be used for this purpose.

The selection of proper basis functions for a particular differential operator is important in the process of deriving the closed-form particular solution. In this paper, we choose the Matern function, also known as the Sobolov function, as the basis function for the derivation of the closed-form particular solution for Helmholtz-type differential equations. Due to the conditioning problems related to the MQ and Gaussian, the Matern function has been considered as an alternative basis function for RBF collocation methods [17–19]. Furthermore, we discover that the Matern function has the desired properties for allowing one to derive the closed-form particular solutions for Helmholtz-type operators. The Matern function is defined as

$$\phi_\nu(r) = \frac{2^{1-\nu}}{\Gamma(\nu)}(cr)^\nu K_\nu(cr) \tag{1}$$

where K_ν is the modified Bessel function of the second kind of order $\nu > 0$ and $c > 0$ is the shape parameter. If ν is of the form $n + 1/2$ where n is a nonnegative integer, then (1) simplifies to

$$\phi_{n+1/2}(r) = \frac{(cr)^n \exp(-cr)}{(2n-1)!!}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!(2cr)^k}. \tag{2}$$

Cheng [13] re-derived the particular solution of Laplace, biharmonic, and the Helmholtz-type equations using polyharmonic splines. The derivation showed that the particular solution can be obtained through the fundamental solution of some differential operators. Inspired by such novel approach, we also use the similar approach to link the particular solution of Helmholtz-type operators to the fundamental solution of certain differential operators. Since the fundamental solutions for various differential operators are well established in the area of the boundary element methods, the derivation of the particular solution can be conducted with little effort.

A good shape parameter is critical to the accuracy. In the RBF literature, various techniques have been developed in selecting a "sub-optimal" shape parameter [20–22]. In this paper, we adopt the so-called Leave-One-Out-Cross-Validation method (LOOCV) [22,23] for automatically selecting a good shape parameter. In this approach, the exact solution is not required in the selection process.

The structure of the paper is as follows. In Section 2, we briefly review the MPS. In Section 3, we derive the closed-form particular solution of Helmholtz-type operators in 2D using the Matern function with integer orders. In Section 4, we make the same derivation in 3D using the Matern function with fractional orders. In Section 5, the derivation of the closed-form particular solution has been extended to the product of Helmholtz-type operators. In Section 6, we perform the numerical test on three examples to demonstrate the stability and high accuracy of the derived particular solutions. Some concluding remarks are placed in the last section.

2. The MPS for Helmholtz-type equations

In this paper, the method of particular solutions (MPS) is used to solve the Helmholtz-type equations as follows:

$$(\Delta \pm \lambda^2)u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{3}$$

with the boundary condition

$$Bu(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{4}$$

where $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, is a bounded and closed domain with boundary $\partial\Omega$, Δ is the Laplacian, B is the boundary operator, and f and g are known functions.

Next, we will briefly review the MPS for solving the above Helmholtz-type problems.

Let $\{\mathbf{x}_i\}_{i=1}^{n}$ be a set of pairwise distinct interpolation points in $\Omega \cup \partial\Omega$, and $\{\mathbf{x}_i\}_{i=1}^{n_i} \subseteq \Omega$, $\{\mathbf{x}_i\}_{i=n_i+1}^{n} \subseteq \partial\Omega$.

In the MPS, we assume $u(\mathbf{x})$ can be approximated by the following linear combination of particular solution Φ

$$u(\mathbf{x}) \simeq \hat{u}(\mathbf{x}) = \sum_{j=1}^{n} a_j \Phi\left(\|\mathbf{x} - \mathbf{x}_j\|\right) \tag{5}$$