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Orthogonal spline collocation method for the fourth-order diffusion system[☆]

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ABSTRACT

The fourth-order diffusion systems depict the wave and photon propagation in intense laser beams and play an important role in the phase separation in binary mixture. In this paper, by using orthogonal spline collocation (OSC) method in spatial direction and classical L1 approximation in temporal direction, a fully discrete scheme is established for a class of fourth-order fractional reaction–diffusion equations. For the original unknown u and auxiliary variable $v = \Delta u$, the full-discrete unconditional stabilities based on a priori analysis are derived by virtue of properties of OSC. Moreover, the convergence rates in L^2-norm for unknown u are strictly investigated. At the same time, the optimal error estimates in H^1-norm for unknown u and in L^2-norm for variable v, are also derived, respectively. For further verifying the theoretical analysis, some numerical examples are provided.

1. Introduction

We consider the two dimensional fourth-order fractional diffusion equation:

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha} = \kappa_1 \Delta u(\mathbf{x},t) - \kappa_2 \Delta^2 u(\mathbf{x},t) + f(\mathbf{x},t), \qquad \mathbf{x} = (x,y) \in \Omega = [0,L] \times [0,L] \subset \mathbb{R}^2,\ t \in (0,T], \tag{1.1}$$

subject to

$$u(\mathbf{x},0) = \varphi(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.2}$$

$$u(\mathbf{x},t) = \Delta u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \partial\Omega \times (0,T], \tag{1.3}$$

where Δ is the Laplace differential operator, $0 < \alpha < 1$, $\kappa_1 \geq 0$ and $\kappa_2 > 0$. The symbol $\frac{\partial^\alpha}{\partial t^\alpha}$ means the Caputo fractional derivative of order α defined by

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(\mathbf{x},s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1. \tag{1.4}$$

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Model (1.1) arises in a variety of applications such as ice formation, fluids in lungs, brain warping and designing special curves on surfaces, phase transition in material science and wave propagation in beams and so on. We refer the reader to [1,2] and the references therein. Up to now, for time-dependent two dimensional fractional evolution equations, the relevant works merely concentrated on problems with second-order space derivative. For example, Fan et al. [3] proposed a novel unstructured mesh FEM for 2D time–space fractional wave equation on irregular convex domain. Zhao et al. [4,5] constructed FEM and classical L1 approximation for 2D fractional diffusion equations, and also obtained convergence and superconvergence results. Zheng et al. [6,7] used a high order spectral method for time-fractional diffusion equations. Zhuang et al. [8] proposed Galerkin FEM for fractional Cable equation, and established strictly error analysis. For more numerical methods for fractional partial differential equations and applications, we refer the reader to [9] and the references therein.

However, the study of numerical methods for fourth-order fractional evolution equations is very little. Some others [10–14] considered one dimensional fourth order fractional sub-diffusion equations by using finite difference method (FDM). Recently, in [15], based on L1 scheme in time and mixed finite element method (FEM) in space, a fully-discrete scheme was proposed for (1.1), which had convergence rate of $O(\tau^{2-\alpha}+\tau^{-\alpha}h^{m+1}+h^{m+1})$ in L^2 norm with respect to the scalar unknown u and $O(\tau^{2-\alpha}+\tau^{-\frac{3\alpha}{2}}h^{m+1}+h^{m+1})$ with respect to the variable $\sigma=\Delta u$, where τ, h and m were the time step, space step and degree of approximate solution, respectively. It is worthwhile to mention that undesirable factors ($\tau^{-\alpha}$ and $\tau^{-\frac{3\alpha}{2}}$) in the space error term grow with decreasing time step, which is not optimal. In [16], we also proposed the quasi-wavelet collocation method for one-dimensional fourth order partial integro-differential equation with a weakly singular kernel. Earlier, mixed methods in combination with OSC methods for classical integer fourth order evolution equations with the first or second order time derivative were proposed in [17–19]. However, OSC methods for two-dimensional fourth-order fractional diffusion equations have not been found in the previous studies.

For elliptic problems, OSC is based on replacing the exact solution by its piecewise polynomial approximation and on satisfying the partial differential equation at the Gauss points (see, e.g., [20] and references therein). The OSC method has evolved as a robust and valuable technique for numerical solutions of PDE [18,20–23]. Recently, Bialecki and Fairweather et al. [24] formulated the extrapolated Crank–Nicolson OSC method with C^1 splines of degree ≥ 3, and established an optimal order error bound in the discrete maximum norm in time and the continuous maximum norm in space. We also formulated OSC and ADI OSC methods to solve fractional PDE [25–28] and give rigorous stability and convergence analysis. In this paper, we examine the use of L1-OSC method in conjunction with a splitting reducing-order procedure for fourth-order fractional diffusion equations (1.1)–(1.3) and derive optimal error estimates. The reason for considering OSC method is that this method is much superior to classical B-splines in terms of stability, efficiency and conditioning of the resulting matrix [22].

The main contribution of the paper consists of the following: Based on the idea of splitting reducing-order procedure, we construct L1-OSC method for fourth-order fractional reaction–diffusion problem (1.1)–(1.3), and provide some two-dimensional numerical examples to verify the effectiveness of proposed methods. Comparisons are made between the present OSC algorithms and existing fourth-order compact FDM in [13], in which we can see that our algorithms are of much better accuracy than the existing one. We also perfect the error estimates in [15] by removing the undesirable factors $\tau^{-\alpha}$ and $\tau^{-\frac{3\alpha}{2}}$ in the space error term. In this paper, the "true" optimal convergence results of L1-OSC method for (1.1) are proved in detail, where the so-called "true" means that mesh size of temporal direction and spatial direction are non-interfering each other in convergence results, by the way, the convergence rates in many papers are not optimal due to the factor $\tau^{-\alpha}$. queryPlease check the sentence 'In this paper ... factor' for clarity, and correct if necessary.

The structure of the rest of this paper is as follows. In Section 2, we first introduce some necessary notations and auxiliary lemmas. Afterward, the L1-OSC method is proposed, and the stability and convergence are analyzed. In Section 3, some numerical results are given to demonstrate the theoretical analysis. The conclusion is made in Section 4.

2. Orthogonal spline collocation (OSC) methods

In this part, we introduce firstly some notations. For positive integers N_x and N_y, let $\delta_x=\{x_i\}_{i=0}^{N_x}$ and $\delta_y=\{y_j\}_{j=0}^{N_y}$ be two partitions of $[0,L]$ such that

$$\delta_x: 0=x_0<x_1<\cdots<x_{N_x}=L, \qquad \delta_y: 0=y_0<y_1<\cdots<y_{N_y}=L.$$

Assume that the partition $\delta=\delta_x\times\delta_y$ of Ω is quasi-uniform [29]. Let $h_k^x=x_k-x_{k-1}$, $h_l^y=y_l-y_{l-1}$, $h=\max(\max_{1\leq k\leq N_x}h_k^x, \max_{1\leq l\leq N_y}h_l^y)$.

Set

$$\mathcal{M}_r(\delta)=\mathcal{M}(r,\delta_x)\otimes\mathcal{M}(r,\delta_y)$$

to be the set of all functions that are finite linear combinations of products of the form $v^x(x)v^y(y)$ where $v^x\in\mathcal{M}(r,\delta_x)$ and $v^y\in\mathcal{M}(r,\delta_y)$. Here,

$$\mathcal{M}(r,\delta_x)=\left\{v|v\in C^1[0,L], v|_{[x_{k-1},x_k]}\in P_r, k=1,2,\ldots,N_x, v(0)=v(L)=0\right\},$$

and P_r denotes the set of polynomials of degree at most r, with $\mathcal{M}(r,\delta_y)$ defined similarly.

Define the set of Gauss collocation points in Ω:

$$\Lambda(\delta)=\{\xi|\xi=(\xi^x,\xi^y), \xi^x\in\Lambda_x, \xi^y\in\Lambda_y\},$$