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Postprocessing Galerkin method using quadratic spline wavelets and its efficiency

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ABSTRACT

The wavelet-Galerkin method is a useful tool for solving differential equations mainly because the condition number of the stiffness matrix is independent of the matrix size and thus the number of iterations for solving the discrete problem by the conjugate gradient method is small. We have recently proposed a quadratic spline wavelet basis that has a small condition number and a short support. In this paper we use this basis in the Galerkin method for solving the second-order elliptic problems with Dirichlet boundary conditions in one and two dimensions and by an appropriate post-processing we achieve the L^2-error of order $\mathcal{O}\left(h^4\right)$ and the H^1-error of order $\mathcal{O}\left(h^3\right)$, where h is the step size. The rate of convergence is the same as the rate of convergence for the Galerkin method with cubic spline wavelets. We show theoretically as well as numerically that the presented method outperforms the Galerkin method with other quadratic or cubic spline wavelets. Furthermore, we present local post-processing for example of the equation with Dirac measure on the right-hand side.

1. Introduction

Let $d \in \mathbb{N}$ and $\Omega = (0,1)^d$. In this paper we focus on the case $d = 1$ and $d = 2$, but all that follows can be generalized. We use the wavelet-Galerkin method to solve the second order elliptic equation

$$-\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left(a_{i,j}(x)\,\frac{\partial u}{\partial x_i}\right)+b(x)\,u=f \text{ on } \Omega, \quad u=0 \text{ on } \partial\Omega. \tag{1}$$

We assume that $b(x) \geq B > 0$, the functions $a_{i,j}$, b and f are sufficiently smooth on $\overline{\Omega}$, and that $a_{i,j}$ satisfy the uniform ellipticity condition

$$\sum_{i=1}^{d}\sum_{j=1}^{d} a_{i,j}(x)\,x_i x_j \geq C\sum_{i=1}^{d} x_i^2, \quad x=(x_1,\ldots,x_d), \tag{2}$$

where $C > 0$ is independent of x. The divergence form of Eq. (1) is

$$-\nabla\cdot(a\nabla u)+bu=f \text{ on } \Omega, \quad u=0 \text{ on } \partial\Omega, \tag{3}$$

where ∇ denotes the standard nabla operator and the matrix a is a matrix of functions $a_{i,j}$. This equation is a basic differential equation and it describes a wide variety of phenomena from physics and engineering. There is already a vast literature on

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the numerical solution of ordinary differential equations and partial differential equations using wavelets, e.g. [1–4]. Many papers are devoted to the wavelet Galerkin method [5–8] and using spline wavelets for solving PDEs [9–11]. However the superconvergence and the efficiency of postprocessing employing superconvergence in the wavelet-Galerkin methods are not widely studied. In the special case of periodical boundary conditions superconvergence for the wavelet-Galerkin method at the nodal points for the initial value problem was proved in [12]. In this paper we review superconvergence results for the elliptic problem (1), we propose postprocessing of the Galerkin method with quadratic spline wavelets and we show theoretically as well as in numerical examples that the wavelet Galerkin method with the proposed postprocessing is more efficient than the wavelet Galerkin methods that realize the same convergence rate, i.e. the Galerkin method with cubic spline wavelets. Therefore, we compare the efficiency of our method and the Galerkin method with cubic spline wavelets, namely wavelets from [13,14,11]. Cubic spline wavelets from [15,16] have the advantage that they have local dual functions which play a role in some applications such as solving nonlinear equations. However, these wavelet bases have larger support and are worse conditioned than wavelets from [13,14,11] and are therefore less efficient. Cubic Hermite spline multiwavelet bases from [17–19] have more vanishing moments but are worse conditioned than a cubic Hermite spline multiwavelet basis from [14] and thus their use in the wavelet-Galerkin method for solving (1) leads to larger number of iterations.

Our motivation is the following: a quadratic spline wavelet basis introduced in Section 2 contains functions with smaller support and the basis is better conditioned than other quadratic spline wavelet bases and also than cubic spline wavelet bases. Therefore the solution of Eq. (1) by the wavelet-Galerkin method using quadratic spline-wavelets requires smaller number of iterations and one iteration requires smaller number of floating point operations than the wavelet-Galerkin method with other quadratic or cubic spline wavelets. If the solution u is sufficiently smooth, then the order of convergence with respect to the L^2-norm for a quadratic basis is $\mathcal{O}\left(h^3\right)$ and for a cubic basis is $\mathcal{O}\left(h^4\right)$. However since the superconvergence occurs in certain points it is possible to use a simple postprocessing to obtain the convergence of order $\mathcal{O}\left(h^4\right)$ also for the quadratic basis. This postprocessing does not increase the computational time, because it is similarly demanding as evaluation of the approximate solution in the given points. Therefore, by postprocessing the Galerkin method with quadratic spline wavelets we obtain method that is more efficient than the Galerkin method with cubic spline wavelets. Recall that classical methods such as finite element method with linear finite elements or finite-difference method are the second order methods with respect to the L^2-norm under assumptions on domain and the smoothness of the solution.

2. Quadratic spline wavelet basis

In [20] we constructed quadratic spline wavelet bases of the spaces $H_0^1(I)$ and $H_0^1(\square)$, where $I = (0, 1)$ and $\square = (0, 1)^2$. We briefly review the construction and main properties of these wavelet bases. We use the bases to solve the problem (3), because scaling functions and wavelets have short support and the condition number of the basis is very small.

First, we briefly recall a definition of a wavelet basis. Let H be a Hilbert space with the norm $\|\cdot\|_H$. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j, k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale*. We define

$$\|\mathbf{v}\|_2 := \sqrt{\sum_{\lambda\in\mathcal{J}} v_\lambda^2}, \quad \text{for} \quad \mathbf{v} = \{v_\lambda\}_{\lambda\in\mathcal{J}},\ v_\lambda \in \mathbb{R},$$

and

$$l^2(\mathcal{J}) := \left\{\mathbf{v} : \mathbf{v} = \{v_\lambda\}_{\lambda\in\mathcal{J}},\ v_\lambda \in \mathbb{R}, \|\mathbf{v}\|_2 < \infty\right\}.$$

Definition 1. A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a *wavelet basis* of H, if

(*i*) Ψ is a *Riesz basis* for H, i.e. the closure of the span of Ψ is H and there exist constants $c, C \in (0, \infty)$ such that

$$c\,\|\mathbf{b}\|_2 \le \left\|\sum_{\lambda\in\mathcal{J}} b_\lambda \psi_\lambda\right\|_H \le C\,\|\mathbf{b}\|_2\,, \tag{4}$$

for all $\mathbf{b} := \{b_\lambda\}_{\lambda\in\mathcal{J}} \in l^2(\mathcal{J})$.

(*ii*) The functions are *local* in the sense that diam *supp* $\psi_\lambda \le C2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, and at a given level j the supports of only finitely many wavelets overlap at any point x.

A scaling basis is defined as a basis of quadratic B-splines in the same way as in [15,21,16]. Let ϕ be a quadratic B-spline defined on knots [0, 1, 2, 3]. It can be written explicitly as:

$$\phi(x) = \begin{cases} \dfrac{x^2}{2}, & x \in [0, 1], \\ -x^2 + 3x - \dfrac{3}{2}, & x \in [1, 2], \\ \dfrac{x^2}{2} - 3x + \dfrac{9}{2}, & x \in [2, 3], \\ 0, & \text{otherwise.} \end{cases}$$