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An upwind compact difference scheme for solving the streamfunction–velocity formulation of the unsteady incompressible Navier–Stokes equation

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ABSTRACT

In this paper, an upwind compact difference method with second-order accuracy both in space and time is proposed for the streamfunction–velocity formulation of the unsteady incompressible Navier–Stokes equations. The first derivatives of streamfunction (velocities) are discretized by two type compact schemes, viz. the third-order upwind compact schemes suggested with the characteristic of low dispersion error are used for the advection terms and the fourth-order symmetric compact scheme is employed for the biharmonic term. As a result, a five point constant coefficient second-order compact scheme is established, in which the computational stencils for streamfunction only require grid values at five points at both (n)th and $(n+1)$th time levels. The new scheme can suppress non-physical oscillations. Moreover, the unconditional stability of the scheme for the linear model is proved by means of the discrete von Neumann analysis. Four numerical experiments involving a test problem with the analytic solution, doubly periodic double shear layer flow problem, lid driven square cavity flow problem and two-sided non-facing lid driven square cavity flow problem are solved numerically to demonstrate the accuracy and efficiency of the newly proposed scheme. The present scheme not only shows the good numerical performance for the problems with sharp gradients, but also proves more effective than the existing second-order compact scheme of the streamfunction–velocity formulation in the aspect of computational cost.

1. Introduction

Over the decades, numerical methods for solving the incompressible Navier–Stokes (N–S) equations have been investigated rather extensively [1–12] (see also references therein). According to the incompressible N–S equations in the different variable formulations, the mathematical formulations of the solutions of the two-dimensional (2D) incompressible equations may be categorized into three types, viz. the primitive variable formulation, the vorticity formulation and the pure stream function or streamfunction–velocity formulation. The primitive variable formulation can represent the fluid flow phenomena accurately, but a significant problem encountered when solving the incompressible N–S equations is that the pressure is coupled implicitly with the divergence-free constraint on the velocity field in the continuity equation. The use of the streamfunction–vorticity or vorticity–velocity formulation may avoid handling the pressure variable. However, when the vorticity transport equation is solved numerically, due to the lack of the simple physical boundary conditions for the vorticity

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field at the no-slip boundaries, a variety of numerical approximations for the boundary values of vorticity has to be carried out. This difficulty can be overcome by using the pure streamfunction formulation [1,6–12] or the streamfunction–velocity formulation [2,3,5,13–17]. The main advantage of this type of formulation is that the boundary conditions of velocity and streamfunction are generally known and are easy to implement computationally, thus the computational schemes proposed are very efficient and accurate.

In this paper, we consider the following streamfunction–velocity formulation of the 2D unsteady N–S equations representing incompressible fluid flows.

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right)+u\frac{\partial}{\partial x}\left(\nabla^2\psi\right)+v\frac{\partial}{\partial y}\left(\nabla^2\psi\right)=\frac{1}{Re}\nabla^4\psi+f,\quad (x,y)\in\Omega$$
$$u=\frac{\partial\psi}{\partial y},\ v=-\frac{\partial\psi}{\partial x} \tag{1}$$

where ψ is the streamfunction, u and v are the velocities and Re is the nondimensional Reynolds number.

For the above streamfunction–velocity formulation of the unsteady incompressible N–S equation, several compact finite difference methods has been proposed in past several years. In 2002, Kupferman [6] presented a two-step second-order central-difference scheme, in which the discretization of the advection term was based on the central-difference scheme with nonlinear slope limiters. For the calculation of the advection term, the inter-function relations between streamfunction and vorticity need to be invoked again in [6], and hence it actually is not strictly based on the streamfunction–velocity formulation. In 2005, Ben-Artzi et al. [9] improved the method to totally avoid the inter-function relations between streamfunction and vorticity, and also proposed a two-step second-order central-difference conditional stable scheme. Kalita and Gupta [13] presented a one-step second-order implicit compact scheme in 2010, in which all of these computational stencils required grid values at nine points at both (n)th and $(n+1)$th time levels. Besides these work, the higher order accurate schemes were proposed by Ben-Artzi et al. [10] in 2010 and the method based on the streamfunction–velocity formulation was also extended in irregular geometries by Sen et al. [14] in 2013. Very recently, Pandit and Karmakar [15] developed an efficient implicit compact scheme, in which all of these computational stencils required grid values at five points at both (n)th and $(n+1)$th time levels. It is noted that the above mentioned methods for discretizing the convective terms are based on central or symmetric-difference schemes [9,10,13–15]. Actually, the use of central or symmetric schemes in convective terms could lead to the non-physical oscillations for the problems with sharp gradients or high Reynolds numbers. Therefore, upwind compact schemes with higher resolution should be considered for the discretization of convective terms. However, to best of our knowledge, no any upwind compact scheme is reported for the streamfunction–velocity formulation of the unsteady incompressible N–S equations.

The purpose of this paper is to established an efficient upwind compact difference algorithm for the streamfunction–velocity formulation of the 2D unsteady incompressible N–S equations. In this work, the first derivatives of streamfunction (velocities) in the advection terms and in the biharmonic term are discretized using the optimized third-order upwind compact schemes with the characteristic of low dispersion error and the fourth-order symmetric compact schemes, respectively. In addition, an efficient five-point compact scheme has been established for the streamfunction–velocity formulation of steady-state incompressible N–S equations [3] and extended to the two dimensional heat transfer [15,18] and MHD problems [19–21]. Thus, we also expect to develop a more efficient scheme for the unsteady case, in which the computational stencils for streamfunction only require grid values at *five* points at both (n)th and $(n+1)$th time levels.

The remainder of this paper is organized as follows. In Section 2, a second-order upwind compact difference algorithm for the streamfunction–velocity formulation of unsteady incompressible N–S equation is proposed using a new derivation approach. Section 3 is devoted to a detailed analysis of stability in a suitable linearized model. Numerical experiments for four test problems are performed to validate the accuracy and efficiency of the newly derived compact difference scheme in Section 4. Conclusions are included in Section 5.

2. Second-order upwind compact difference approximation

2.1. Spatial discretization

In this section we formulate a second-order upwind compact FD method that can solve Eq. (1). The compact method for the discretization of the streamfunction formulation uses not only the values of unknown variable but also the values of its first derivatives at selected grid points.

For brief sake, we number the nine mesh points (x,y), $(x+h,y)$, $(x,y+h)$, $(x-h,y)$, $(x,y-h)$, $(x+h,y+h)$, $(x-h,y+h)$, $(x-h,y-h)$ and $(x+h,y-h)$ as 0, 1, 2, 3, 4, 5, 6, 7 and 8 respectively (see Fig. 1), where h denotes the space mesh size of x and y directions.

And then, some standard finite difference operators at the grid point (x,y) are defined as follows:

$$\delta_x^2\delta_y\phi=\frac{\phi_5+\phi_6-\phi_7-\phi_8-2(\phi_2-\phi_4)}{2h^3},$$
$$\delta_x\delta_y^2\phi=\frac{\phi_5-\phi_6-\phi_7+\phi_8-2(\phi_1-\phi_3)}{2h^3},$$