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Infinitely many solutions and least energy solutions for Klein–Gordon–Maxwell systems with general superlinear nonlinearity☆

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A B S T R A C T

This paper is concerned with the following Klein–Gordon–Maxwell system:

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x, u), & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\omega > 0$ is a constant, $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, $f \in \mathcal{C}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, and f is superlinear at infinity. Using some weaker superlinear conditions instead of the common super-cubic conditions on f, we prove that the above system has (1) infinitely many solutions when $V(x)$ is coercive and sign-changing; (2) a least energy solution when $V(x)$ is positive periodic. These results improve the related ones in the literature.

1. Introduction

In this paper, we study the following Klein–Gordon–Maxwell system:

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x, u), & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\omega > 0$ is a constant, $u, \phi : \mathbb{R}^3 \to \mathbb{R}$, $V : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. This type of system was first introduced by Benci and Fortunato [1] as a model describing solitary waves for the nonlinear Klein–Gordon equation interacting with an electromagnetic field. The presence of the nonlinear term f simulates the interaction between many particles or external nonlinear perturbations. For more details in the physical aspects, we refer the readers to [1,2].

In the past decades, there has been many existence and nonexistence results on (1.1) with constant potential $V(x) = m_0^2 - \omega^2$. By variational method, Benci and Fortunato [2] first proved that the following special form of (1.1)

$$\begin{cases} -\Delta u + \left[m_0^2 - (\omega + \phi)^2\right] u = |u|^{q-2}u, & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3 \end{cases} \tag{1.2}$$

possesses infinitely many radially symmetric solutions when $0 < \omega < m_0$ and $4 < q < 6$. Afterwards, D'Aprile and Mugnai [3] obtained the same conclusion when $0 < \omega < \sqrt{(q-2)/2}\, m_0$ and $2 < q \le 4$. Based on a Pohozaev-type

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argument, D'Aprile and Mugnai [4] showed that (1.2) only has a trivial solution when $0 < q \le 2$ or $q \ge 6$. Later, inspired by [2,3], by some a priori estimates, Azzollini and Pomponio [5] proved that (1.2) admits a least energy solution, namely a solution which minimizes the energy functional among all nontrivial solutions, if one of the following conditions holds:
(i) $4 \le q < 6$ and $0 < \omega < m_0$;
(ii) $2 < q < 4$ and $0 < \omega < \sqrt{g_1(q)}\, m_0$, where $g_1(q) = (q-2)/(6-q)$.
Their method consisted in minimizing the corresponding functional associated with (1.2) on the Nehari manifold. Following the ideas of [5], Wang [6] further improved the existence range of (m_0, ω) for $2 < q < 4$ as follows:

$$0 < \omega < \sqrt{g_2(q)}\, m_0 \ \text{ with } \ g_2(q) = \frac{4(q-2)}{(4-q)^2 + 4(q-2)}. \tag{1.3}$$

By combining an indirect method, developed by Struwe [7] and Jeanjean [8], with the Pohozaev identity, established by Cassani [9] and D'Aprile and Mugnai [3], Azzollini et al. [10] proved that (1.2) has a nontrivial radial solution under a weaker condition than (1.3) when $2 < q < 4$. Recently, the results in [10] were improved by Chen and Tang [11]. Some related results for (1.2) can be found in [9,12,13] and the references therein. It is worth mentioning that the approaches used in [5,10,11,6] rely heavily on the differentiability of nonlinear terms or the Pohozaev identity, which are no longer applicable for (1.1) with $f \in \mathcal{C}$.

For (1.1) with non-constant potential, via a variant fountain theorem, He [14] first proved the existence of infinitely many solutions under the following assumptions:

(V1′) $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, $\inf_{\mathbb{R}^3} V > 0$ and there exists a constant $r > 0$ such that

$$\lim_{|y|\to\infty} \operatorname{meas}\{x \in \mathbb{R}^3 : |x-y| \le r,\ V(x) \le M\} = 0, \quad \forall\, M > 0;$$

(F1) $f \in \mathcal{C}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist constants $C_0 > 0$ and $p \in (2, 6)$ such that

$$|f(x,t)| \le C_0 \left(|t| + |t|^{p-1}\right), \quad \forall\, (x,t) \in \mathbb{R}^3 \times \mathbb{R};$$

(F2) $f(x,t) = -f(x,-t)$, $\forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}$;
(AR) there exists $\nu > 4$ such that

$$\frac{1}{\nu} f(x,t)t \ge F(x,t) := \int_0^t f(x,s)\mathrm{d}s, \quad \forall\, (x,t) \in \mathbb{R}^3 \times \mathbb{R};$$

or

(AR′) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^3} = +\infty$ uniformly in $x \in \mathbb{R}^3$, and $f(x,t)t - 4F(x,t) \to \infty$ as $|t| \to \infty$ uniformly in $x \in \mathbb{R}^3$.

As we know, (AR) or (AR′) is a very convenient hypothesis, it readily achieves mountain pass geometry and shows the boundedness of the Palais–Smale or Cerami sequences. Subsequently, by the symmetric mountain pass theorem, Ding and Li [15] and Li and Tang [16] weakened (V1′), (AR) and (AR′) to the following weaker conditions:

(V1) $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, $\inf_{\mathbb{R}^3} V > -\infty$ and there exists a constant $r > 0$ such that

$$\lim_{|y|\to\infty} \operatorname{meas}\{x \in \mathbb{R}^3 : |x-y| \le r,\ V(x) \le M\} = 0, \quad \forall\, M > 0;$$

(SF) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^4} = +\infty$ uniformly in $x \in \mathbb{R}^3$;
(SF′) there exists $\theta_1 \ge 0$ such that

$$f(x,t)t - 4F(x,t) + \theta_1 t^2 \ge 0, \quad \forall\, (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

When $V(x)$ is sign-changing, as far as we know, it seems to be necessary that $F(x,t)$ is of 4-superlinear growth at $t = \infty$ in the literature.

Motivated by the above works and [17], the first object of the present paper is to generalize and improve the results obtained in [15,14,16] by relaxing (AR), (AR′), (SF) and (SF′) to the following conditions:

(F3) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^2} = +\infty$ uniform in $x \in \mathbb{R}^3$, and there exists $r_0 > 0$ such that

$$F(x,t) \ge 0, \quad \forall\, x \in \mathbb{R}^3,\ |t| \ge r_0;$$

(F4) there exist $\mu > 2$ and $\theta > 0$ such that

$$f(x,t)t - \mu F(x,t) + \theta t^2 \ge 0, \quad \forall\, (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$