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Exploring high-order three dimensional virtual elements: Bases and stabilizations

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ABSTRACT

We present numerical tests of the virtual element method (VEM) tailored for the discretization of a three dimensional Poisson problem with high-order "polynomial" degree (up to $p = 10$). Besides, we discuss possible reasons for which the method could return suboptimal/wrong error convergence curves. Among these motivations, we highlight ill-conditioning of the stiffness matrix and not particularly "clever" choices of the stabilizations. We propose variants of the definition of face/bulk degrees of freedom, as well as of stabilizations, which lead to methods that are much more robust in terms of numerical performances.

Darest thou now, O Soul,
Walk out with me toward the Unknown Region,
Where neither ground is for the feet, nor any path to follow?
Walt Whitman, Leaves of Grass, 1855.

1. Introduction

The virtual element method (VEM) is a generalization of the finite element method (FEM) that allows for general polytopal meshes, thus including non-convex elements and hanging nodes.

Approximation spaces in VEM contain locally polynomials and, more in general, consist of functions which solve local problems mimicking the original ones and, consequently, are not known in a closed form (hence the name virtual). For this reason, the operators involved in the discretization of the problem are not computed exactly; rather, the construction of the method is based on two ingredients: proper projectors onto piecewise discontinuous polynomial spaces and stabilizing bilinear forms mimicking their continuous counterparts. Both ingredients can be computed exactly only with the aid of the degrees of freedom.

Although the VEM technology is very recent, it has been applied to a large number of two dimensional problems; a short list of them is: [1–7]; in particular, high-order VEM are investigated in [8–13].

The literature dealing with three dimensional problems is much less broad, see [14–16]. The only attempt, to the best of our knowledge, to increase the order of VEM in 3D is [17], where the highest order achieved in numerical tests is $p = 5$.

In the present work, we have a double aim. Firstly, we present numerical tests for three dimensional VEM of order higher than 5, thus inviting the reader to raise the anchor from the safe port [17] and to "…*Walk out with us toward the Unknown*

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Region… ", reaching in fact the pinnacle of degree of accuracy $p = 10$. Secondly, we numerically investigate the reasons of possible suboptimal/wrong behaviour in the error convergence curves, highlighting two among them: the ill-conditioning of the linear system stemming from the method and the choice of the stabilization.

We tackle the (possible) issue of suboptimality of VEM, when considering its h and p versions as well as when it is applied to meshes with elements having collapsing bulk, by proposing two novel approaches of the definition of the face/bulk degrees of freedom and proposing three different stabilizations.

The outline of the paper follows. In Section 2, we review the construction of three dimensional VEM, emphasizing in particular different stabilizations and face/bulk degrees of freedom. Next, in Section 3, we provide a number of numerical results comparing the effects on the method of the above-mentioned stabilizations and degrees of freedom; more precisely, we study the h and the p versions of the method, paying attention also to VEM applied to meshes with degenerate elements. Concluding remarks are stated in Section 4. Finally, in Appendix, we give some hints regarding the implementation of the method with the novel canonical basis functions.

Notation. By $\mathbb{P}_p(F)$ and $\mathbb{P}_p(K)$, $p \in \mathbb{N}$, we denote the spaces of two and three dimensional polynomials of degree p over a polygon F and a polyhedron K, respectively; if $p = -1$, then we set $\mathbb{P}_{-1}(F) = \mathbb{P}_{-1}(K) = \emptyset$. Moreover, we fix:

$$n_p^F = \dim(\mathbb{P}_p(F)), \qquad n_p = \dim(\mathbb{P}_p(K)) \quad \forall p \in \mathbb{N}. \tag{1}$$

Assume now that we are given $\{m_\alpha\}_{\alpha=1}^{n_p}$, $p \in \mathbb{N}$, a basis of $\mathbb{P}_p(K)$ such that:

$$\text{span}\left(\{m_\alpha\}_{\alpha=1}^{n_{p-2}}\right) = \mathbb{P}_{p-2}(K) \quad \text{and} \quad \text{span}\left(\{m_\alpha\}_{\alpha=1}^{n_{p-1}}\right) = \mathbb{P}_{p-1}(K). \tag{2}$$

It will be convenient to split the polynomial basis into:

$$\{m_\alpha\}_{\alpha=1}^{n_p} = \{m_\alpha\}_{\alpha=1}^{n_{p-2}} \cup \{m_\alpha\}_{\alpha=n_{p-2}+1}^{n_{p-1}} \cup \{m_\alpha\}_{\alpha=n_{p-1}+1}^{n_p}. \tag{3}$$

We assume that the polygonal counterpart of (2) holds true; consequently, we can consider a splitting analogous to the one in (3) on

$$\mathbb{P}_p(F) = \text{span}\left(\{m_\alpha^F\}_{\alpha=1}^{n_p^F}\right),$$

the space of polynomial of degree p over polygon F.

Remark 1. In the remainder of the paper, we assume the following:

$$\text{the polynomial bases henceforth employed are invariant with respect to homothetic transformations.} \tag{4}$$

2. VEM: definition, stabilizations and bases

In this section, we introduce a family of VEM tailored for the approximation of the following Poisson problem in three dimensions with (for simplicity) homogeneous boundary conditions. Given $\Omega \subset \mathbb{R}^3$ a polyhedral domain and $f \in L^2(\Omega)$:

$$\begin{cases} \text{find } u \in V \text{ s. th.} \\ a(u, v) = (f, v)_{0,\Omega} \quad \forall v \in V, \end{cases} \tag{5}$$

where:

$$V = H_0^1(\Omega), \qquad a(\cdot, \cdot) = (\nabla \cdot, \nabla \cdot)_{0,\Omega}. \tag{6}$$

In Section 2.1, we briefly recall from [1,17,18] the construction of three dimensional VEM for the approximation of the solution to problem (5), keeping yet at a very general level the definition of the degrees of freedom and of the stabilization of the method, typical of the VEM framework. Various choices of stabilizations as well as of face/bulk degrees of freedom are investigated in Sections 2.2 and 2.3, respectively.

2.1. A family of VEM

In this section, we introduce, following [1,17], a family of VEM in three dimensions for the approximation of problem (5).

The VEM in three dimensions is based on conforming sequences $\mathcal{T}_n$ of polyhedra partitioning the physical domain Ω of the PDE of interest. By conforming sequence, we mean that, given $\mathcal{F}_n$, $\mathcal{E}_n$ and $\mathcal{V}_n$ the sets of all faces, edges and vertices of the polyhedra in $\mathcal{T}_n$, respectively, then all the internal faces $F \in \mathcal{F}_n$ must belong to the intersection of two polyhedra.

We observe that, since the aim of the present paper is to test the robustness of the method to mesh-distortion and to increasing "polynomial degrees", no particular geometrical assumptions on the mesh are demanded.