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The Drazin inverse of an even-order tensor and its application to singular tensor equations

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ABSTRACT

The notion of the Moore–Penrose inverses of matrices was recently extended from matrix space to even-order tensor space with Einstein product in the literature. In this paper, we further study the properties of even-order tensors with Einstein product. We define the index and characterize the invertibility of an even-order square tensor. We also extend the notion of the Drazin inverse of a square matrix to an even-order square tensor. An expression for the Drazin inverse through the core-nilpotent decomposition for a tensor of even-order is obtained. As an application, the Drazin inverse solution of the singular linear tensor equation $\mathcal{A} * \mathcal{X} = \mathcal{B}$ will also be included.

1. Introduction

Due to the practical use of high dimensional tensors in many fields such as chemometrics, computer vision, data mining, signal processing, and graph analysis etc. [1–7], the research on tensors has been very active recently [8–16].

For a positive integer N, let $[N] = \{1, 2, \ldots, N\}$. An order k tensor $\mathcal{A} = (A_{i_1 \ldots i_k}) \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_k}$ is a multidimensional array with $I_1 I_2 \ldots I_k$ entries over complex field $\mathbb{C}$, where $i_j \in [I_j], j \in [k]$. Given $\mathcal{A} = (A_{i_1 \ldots i_k})$ and $\mathcal{B} = (B_{i_1 \ldots i_k}) \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_k}$ and a scalar $\alpha \in \mathbb{C}$, with the standard addition $\mathcal{A} + \mathcal{B} = (A_{i_1 \ldots i_k} + B_{i_1 \ldots i_k})$ and the scalar product $\alpha A = (\alpha A_{i_1 \ldots i_k})$, $\mathbb{C}^{I_1 \times I_2 \times \cdots \times I_k}$ is a vector space. The vector space $\mathbb{C}^n$ and matrix space $\mathbb{C}^{I_1 \times I_2}$ are two special examples of tensor spaces.

For tensors $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_k \times J_1 \times \cdots \times J_k}$ and $\mathcal{B} \in \mathbb{C}^{J_1 \times \cdots \times J_k \times J_{k+1} \times \cdots \times J_m}$ with $m \geq k$, the Einstein product $A *_k B$ of tensors $\mathcal{A}$ and $\mathcal{B}$ is a tensor in $\mathbb{C}^{I_1 \times \cdots \times I_k \times J_{k+1} \times \cdots \times J_m}$ defined in [17] by

$$(\mathcal{A} *_k \mathcal{B})_{i_1 \ldots i_k j_{k+1} \ldots j_m} = \sum_{j_r \in [J_r], r \in [k]} A_{i_1 \ldots i_k j_1 \ldots j_k} B_{j_1 \ldots j_k j_{k+1} \ldots j_m}. \tag{1.1}$$

This tensor product satisfies the associative law. When $m = k$, $\mathcal{A} *_k \mathcal{B}$ is in $\mathbb{C}^{I_1 \times \cdots \times I_k}$ for tensor $\mathcal{B} \in \mathbb{C}^{J_1 \times \cdots \times J_k}$. Thus, the $2k$-order tensor $\mathcal{A}$ can be viewed as an operator from tensor space $\mathbb{C}^{J_1 \times \cdots \times J_k}$ to tensor space $\mathbb{C}^{I_1 \times \cdots \times I_k}$. Let this operator be denoted by $L_{\mathcal{A}}$. $L_{\mathcal{A}}$ is indeed a linear operator between two tensor spaces of order-k. For simplicity, we will not distinguish the difference between $L_{\mathcal{A}}$ and $\mathcal{A}$ and omit the subindex k, i.e., $L_{\mathcal{A}}(\mathcal{X}) = \mathcal{A} *_k \mathcal{X} \equiv \mathcal{A} * \mathcal{X}$.

Define the inner product on $\mathbb{C}^{N_1 \times \cdots \times N_k}$

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \sum_{n_r \in [N_r], r \in [k]} \overline{\mathcal{X}}_{n_1 \ldots n_k} \mathcal{Y}_{n_1 \ldots n_k}$$

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for any $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{N_1\times\cdots\times N_k}$ and the Frobenius norm $\|\cdot\|_F$ is defined as $\|\mathcal{X}\|_F = \sqrt{\langle\mathcal{X},\mathcal{X}\rangle}$. Obviously, we have (1) $\langle\mathcal{X},\mathcal{Y}\rangle = \overline{\langle\mathcal{Y},\mathcal{X}\rangle}$; (2) $\|\mathcal{X}\|_F = 0$ if and only if $\mathcal{X} = 0$; and (3) $\|\mathcal{X}+\mathcal{Y}\|_F^2 = \|\mathcal{X}\|_F^2 + \langle\mathcal{X},\mathcal{Y}\rangle + \langle\mathcal{Y},\mathcal{X}\rangle + \|\mathcal{Y}\|_F^2$. For a scalar α in $\mathbb{C}$, $\bar{\alpha}$ is the conjugate of α. For a tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$, the conjugate transpose $\mathcal{A}^*$ of $\mathcal{A}$ is defined by $(\mathcal{A}^*)_{i_1\ldots i_k j_1\ldots j_k} = \bar{\mathcal{A}}_{j_1\ldots j_k i_1\ldots i_k}$. A simple calculation indicates that

$$\langle\mathcal{W}*\mathcal{X},\mathcal{Y}\rangle = \langle\mathcal{X},\mathcal{W}^**\mathcal{Y}\rangle \tag{1.2}$$

for any $\mathcal{W} \in \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$, $\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_k}$, and $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_k}$ [18].

The diagonal tensor $\mathcal{D}$ in $\mathbb{C}^{N_1\times\cdots\times N_k\times N_1\times\cdots\times N_k}$ is the tensor with its entries defined by

$$(\mathcal{D})_{i_1\ldots i_k j_1\ldots j_k} = \begin{cases} d_{i_1\cdots i_k} & \text{if } i_r = j_r \in [N_r], \text{ for } r \in [k] \\ 0 & \text{otherwise,} \end{cases}$$

where $d_{i_1\cdots i_k}$ is a complex number. If all the diagonal entries $d_{i_1\cdots i_k} = 1$, then the diagonal tensor $\mathcal{D}$ is called the identity tensor, denoted by $\mathcal{I}$. The identity tensor depends on the dimensions $N_1, N_2, \ldots, N_k$ of all orders. For simplicity, we will not indicate its dependency on the dimension of each order and use $\mathcal{I}$ to denote both identity tensors in $\mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$ and $\mathbb{C}^{J_1\times\cdots\times J_k\times J_1\times\cdots\times J_k}$. It is easy to show that $(\mathcal{A}*\mathcal{B})^* = \mathcal{B}^**\mathcal{A}^*$ and $\mathcal{I}*\mathcal{A} = \mathcal{A}*\mathcal{I} = \mathcal{A}$ for $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$ and $\mathcal{B} \in \mathbb{C}^{J_1\times\cdots\times J_k\times L_1\times\cdots\times L_k}$ [19].

For a tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$, if there exists a tensor $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$ such that $\mathcal{A}*\mathcal{B} = \mathcal{B}*\mathcal{A} = \mathcal{I}$, then $\mathcal{A}$ is said to be invertible and the $\mathcal{B}$ is called the inverse of $\mathcal{A}$ and denoted by $\mathcal{A}^{-1}$ [20]. For a general tensor $\mathcal{A}$ in $\mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$, its inverse may not exist. But it is shown in [19] that there exists a unique $\mathcal{X}$ in $\mathbb{C}^{J_1\times\cdots\times J_k\times I_1\times\cdots\times I_k}$ satisfying

$$\mathcal{A}*\mathcal{X}*\mathcal{A} = \mathcal{A} \tag{1.3}$$

$$\mathcal{X}*\mathcal{A}*\mathcal{X} = \mathcal{X} \tag{1.4}$$

$$(\mathcal{A}*\mathcal{X})^* = \mathcal{A}*\mathcal{X} \tag{1.5}$$

$$(\mathcal{X}*\mathcal{A})^* = \mathcal{X}*\mathcal{A}. \tag{1.6}$$

The unique $\mathcal{X}$, denoted by $\mathcal{A}^\dagger$, is called the Moore–Penrose inverse of $\mathcal{A}$. Obviously, we have $\mathcal{A}^\dagger = \mathcal{A}^{-1}$ if $\mathcal{A}$ is invertible. The weighted Moore–Penrose inverse of an even-order tensor was recently introduced in [18]. Many results on the generalized inverses, the $\mathcal{X}$ which only satisfies some of the four Eqs. (1.3)–(1.6), can be found in [21,22]. The Moore–Penrose inverses of tensors with an alternative product and the application to linear models are documented in [23].

In this paper, we further develop useful and important properties of even-order tensors with Einstein product. We show that the dimension of $R(\mathcal{A})$ is equal to that of $R(\mathcal{A}^*)$ which, together with the fundamental theorem of even-order tensors [18], enables us to define the index and the Drazin inverse of a tensor in $\mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$. An explicit expression for the Drazin inverse of a tensor is obtained through its core-nilpotent decomposition. As an application, the Drazin inverse solution [24–27] of the singular linear tensor equation $\mathcal{A}*\mathcal{X} = \mathcal{B}$ will also be presented at the end of the paper.

2. Further properties of even-order tensors

In this section, we collect a few useful properties of even-order tensors.

Define the null space and the range of $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$ to be

$$N(\mathcal{A}) = \{\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_k} : \mathcal{A}*\mathcal{X} = 0\} \text{ and } R(\mathcal{A}) = \{\mathcal{A}*\mathcal{X} : \mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_k}\}$$

respectively. It is easily seen that $N(\mathcal{A})$ is a subspace of $\mathbb{C}^{J_1\times\cdots\times J_k}$ and $R(\mathcal{A})$ is a subspace of $\mathbb{C}^{I_1\times\cdots\times I_k}$. The null spaces and ranges of the tensors $\mathcal{A}^*$ and $\mathcal{A}^**\mathcal{A}$ can be defined similarly. The orthogonal complement of a subspace $\mathcal{L}$ in $\mathbb{C}^{N_1\times\cdots\times N_k}$ is defined by

$$\mathcal{L}^\perp = \left\{\mathcal{X} \in \mathbb{C}^{N_1\times\cdots\times N_k} : \langle\mathcal{X},\mathcal{Y}\rangle = 0 \text{ for all } \mathcal{Y} \in \mathcal{L}\right\}.$$

Lemma 2.1. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$. Then*

(1) $N(\mathcal{A})^\perp = R(\mathcal{A}^*)$, $R(\mathcal{A})^\perp = N(\mathcal{A}^*)$, $R(\mathcal{A}^*)^\perp = N(\mathcal{A})$, *and* $N(\mathcal{A}^*)^\perp = R(\mathcal{A})$;
(2) $N(\mathcal{A}^**\mathcal{A}) = N(\mathcal{A})$ *and* $R(\mathcal{A}^**\mathcal{A}) = R(\mathcal{A}^*)$.

Lemma 2.1 was established under a more general setting in [18]. When $k = 1$, it reduces to the fundamental theorem of linear algebra [28]. Lemma 2.1 has played a key role in establishing an alternative proof, which is much simpler than the one in [19], for the minimum-norm and least-squares solution to the tensor equation $\mathcal{A}*\mathcal{X} = \mathcal{B}$. It is this result that helps us to define the index of a "square" tensor, which paves the way to define the Drazin inverses of even-order tensors.

Denote the dimension of a subspace $\mathcal{L}$ by $\dim(\mathcal{L})$. For a matrix $A \in \mathbb{C}^{m\times n}$, we have $\dim(R(A)) = \dim(R(A^*))$. For tensors, we have the following similar result.

Lemma 2.2. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$. Then, we have* $\dim(R(\mathcal{A})) = \dim(R(\mathcal{A}^*))$.

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