ARTICLE IN PRESS
Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A mixed finite element formulation for slightly compressible finite elasticity with stiff fibre reinforcement. Two fibre families. Uniaxial tension formulation

Adam Zdunek [a,*], Waldemar Rachowicz [b]

[a] HB BerRit, Solhemsbackarna 73, SE-163 56 Spånga, Sweden
[b] Institute for Computer Modelling, Section of Applied Mathematics, Cracow University of Technology, Pl 31-155 Cracow, Poland

ARTICLE INFO

Article history:
Received 12 August 2017
Received in revised form 17 December 2017
Accepted 22 December 2017
Available online xxxx

Keywords:
Anisotropic
Two fibres
Hyperelasticity
Inextensible
Finite element
hp-adaptivity

ABSTRACT

We propose a novel finite element based formulation for the solution of the static mechanical mixed boundary value problem of a finite elastic solid reinforced by two distinct, stiff fibre families. The fibre tensions, are assumed decoupled and uniaxial, at out-set. The associated energy conjugate fibre stretch rates are shown to be uniaxial by duality. The natively displacement dependent fibre tension–fibre stretch pairs are replaced by auxiliary independent variables. The complementary, displacement based, stresses and energy conjugate strain rates become tensionless and stretch-rate-less in the two fibre directions, respectively, by construction. An additively decoupled hyperelastic strain energy ansatz in terms of the fibre stretches and a novel apparently doubly stretchless Cauchy–Green tensor is used. The displacement based part of the formulation is set in an apparently inextensible fibre metric space. The proposed uniaxial fibre tension description is statically exact for the fully constrained problem, and the novel doubly stretchless Cauchy–Green tensor is conditionally kinematically admissible in its vicinity. The formulation is realised as a five-field mixed finite element method admitting separate higher order approximations in H^1, for the displacement, and in L^2, for the energy conjugate fibre tensions and stretches, respectively. The convergence and correctness of the implementation is verified by numerical and analytical examples.

1. Introduction

Strongly anisotropic finite elastic materials play an essential role in fields including the mechanics of fibre reinforced rubber-like materials, soft biological tissues and in many other cases. The common source of anisotropy is the high stiffness contrast between the reinforcing fibres and the surrounding matrix material. Continuum models are popular. They capture the strong anisotropy using the concept of preferred directions. In the case of soft tissues the strong contrast is due to the collagen fibres which often show exponential stiffening already at moderate extensions. This phenomenon evolves into near inextensibility with increasing extension. The limiting fully inextensible problem is not a part of the standard pure displacement formulation, due to the reactive uniaxial fibre tensions. Approaching it numerically may cause severe convergence problems, similar to the volumetric locking in near incompressibility. The standard remedy for near-incompressible finite elasticity involves the well known Flory [1] split of the deformation gradient. It is used in the popular displacement, pressure and dilatation three-field $(\boldsymbol{U}, \tilde{p}, \tilde{J})$ finite element (FE) formulation due to Simo, Taylor and Pister (STP) [2]. This mixed formulation with adequate approximations in $[H^1(\Omega)]^3$ and $L^2(\Omega)$ FE-spaces $(\mathscr{V}_p/\mathscr{Q}^2_{p-1})$

* Corresponding author.
E-mail addresses: adam.zdunek@berrit.se (A. Zdunek), wrachowicz@pk.edu.pl (W. Rachowicz).

https://doi.org/10.1016/j.camwa.2017.12.020
0898-1221/

for the displacement $\boldsymbol{U}$ and the auxiliary variables $(\tilde{p}, \tilde{J})$, respectively, provides an effective FE-modelling of isotropic nearly-incompressible rubber-like hyperelasticity. In Zdunek and Rachowicz [3,4], we extended the mixed STP [2] isotropic formulation for reinforcement with one family of stiff fibres introducing additional auxiliary variables for fibre tension $\tilde{\varrho}$ and fibre stretch $\tilde{\lambda}$. There we developed and tested higher order five-field $(\boldsymbol{U}, \tilde{p}, \tilde{J}, \tilde{\varrho}, \tilde{\lambda})$ element constructs $(\mathscr{V}_p/\mathscr{Q}^4_{p-1})$, appropriate mixed residual based *a posteriori* error estimation and we applied an h-adaptive mesh refinement. Strong anisotropy has continued to draw our attention [5,6], and recently attracted others attention as well, see Schröder et al. [7]. It is known that FE polynomial enrichments, $p \geq 4$, restore a uniform convergence when approaching the incompressible limit. The generalisation of this recipe to inextensibility is however numerically contradicted in Zdunek and Rachowicz [6]. In other words, approaching the limiting inextensible problem numerically with a pure displacement based theory may lead to loss of convergence, even with a higher order polynomial ansatz, $p \geq 4$. This novel numerical finding underlines the need to use mixed methods for strong anisotropy.

Further, the assumption of near-incompressibility for soft tissue, in particular for arteries, is currently questioned on experimental grounds by Yosibash and collaborators, [8,9]. Adopting this important shift of the paradigm we take into account that a slight compressibility in combination with a possible inextensibility makes the dilatation in the Flory split [1] kinematically inadmissible, see Zdunek and Rachowicz [5]. This observation has far reaching consequences, and it is related to the problems reported by Vergori et al. [10]. These and related anomalies are addressed earlier by Sansour [11] and recently by Nolan et al. [12]. The remedies proposed in [11] and [12] use the Flory split [1] selectively in the isotropic part of strain energy ansatz but not in its anisotropic part. A related particular element construct with simplified kinematics for strong anisotropy (SKA) is considered by Schröder et al. [7]. In the described constrained situation $(\tilde{\lambda} = 1, \tilde{J} \neq 1)$ we are obliged to bypass the Flory split all together. In other words, we do not use the common decoupled material description for near-incompressibility in terms of the dilatation and volume preserving stretch tensor $\{J, \bar{\mathbf{C}}\}$ but use a description for slightly but finite compressibility. The auxiliary variables in the mixed method concern the fibre stretches and the energy conjugate fibre tensions. Using a mixed 3-field $(\boldsymbol{U}, \tilde{\varrho}, \tilde{\lambda})$ formulation with hierarchical elements of the type $\mathscr{V}_p/\mathscr{Q}^2_{p-1}$ for compressible finite hyperelasticity we resolved the problem with the isostatic load [10,12]. Further, we extended our residual based *a posteriori* error estimation for the compressible 3-field formulation and illustrated an h-adaptive mesh refinement applied to the pressurisation of the saccular- and fusiform aneurysm-like geometries, in [5] and [6] respectively.

Our previous works [3,4,6,5] were devoted to reinforcement by one rapidly stiffening family of fibres under extension. There we started in kinematics deriving a generalised right Cauchy–Green tensor by exchanging the inherent displacement based fibre stretch by an auxiliary independent stretch variable. The associated fibre stretchless right Cauchy–Green tensor was then obtained by setting the auxiliary fibre stretch to unity. The approach here is fundamentally different. It is based on statics. Realising that the sum of two simple uniaxial tensions is a projection we split the stress into fibre tension and a complementary fibre tensionless stress. Further we consider reinforcement by two families of fibres, which stiffen rapidly under extension. This kind of behaviour is observed in arterial and cardiac mechanics where two preferred directions have to be used. Two distinct families are frequently assumed mechanically equivalent. The field has a rich literature. Much remains, however, to be done in finite hyperelasticity concerning the theory and a formulation which is stable in the inf–sup [13] sense for the vicinity of the doubly inextensible problem, see Fosdick and MacSithigh [14] and references therein. Criscione and Hunter [15] developed a framework for extensible continuum finite elasticity for two family fibre reinforcement using the local orthogonal frame obtained by bisecting the material fibre directions. We develop our theory in the material fibre triad. It yields new insights, notably to the point-wise shear between the fibre families, i.e. to the stress response due to the change of angle between the preferred directions. This part of the response is seldom considered.

The remainder of the paper is organised as follows. The description of the kinematics in the fibre triad is presented in Section 2. The basic assumption we use concerns the stress split into superposed uniaxial fibre tensions and a complementary fibre tensionless stress is stated in Section 3. An apparently doubly stretchless right Cauchy–Green tensor which is conditionally positive definite is derived by energy conjugate analysis in Section 4. The mixed hyperelastic constitutive formulation is developed in Section 5. The simple model material is set in Section 6. The mixed five field $(\boldsymbol{U}, \tilde{\varrho}^{\mathrm{F}}, \tilde{\lambda}_{\mathrm{F}})$,[1] $\mathrm{F} = 1, 2$, Hu–Washizu type FE-formulation is developed in Section 7. The associated mixed element constructs of the type $(\mathscr{V}_p/\mathscr{Q}^4_{p-1})$ are briefly described in Section 7.2. The model examples verifying the correctness of implementation of the FE-formulation are presented in Section 8. The paper is finalised by a short discussion in Section 9 and summarised and concluded in Section 10.

Notation. Vectors are denoted by boldface italic lower and upper case letters, e.g. $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ respectively. Co-vectors are denoted with an underset tilde, e.g. $\underset{\sim}{\boldsymbol{a}}$, $\underset{\sim}{\boldsymbol{b}}$, $\underset{\sim}{\boldsymbol{c}}$ and $\underset{\sim}{\boldsymbol{A}}$, $\underset{\sim}{\boldsymbol{B}}$, $\underset{\sim}{\boldsymbol{C}}$ respectively. Second order tensors are denoted by boldface Roman or Greek lower and upper case letters, e.g. $\boldsymbol{\tau}$, $\mathbf{a}$, $\underset{\sim}{\mathbf{g}}$ and $\underset{\sim}{\mathbf{S}}$, $\mathbf{A}$, $\mathbf{G}$. Fourth order tensors are denoted by blackboard bold lower and upper case letters, e.g. $\mathbb{p}$ and $\mathbb{P}$ respectively. The contraction, duality-pairing, of vectors and co-vectors on a given vector space $\mathscr{V} = (\mathscr{F}; \mathbb{R})$, is denoted as,

$$\langle \bullet, \bullet \rangle_{\mathscr{V}} \; : \; \mathscr{V}^* \times \mathscr{V} \to \mathbb{R}, \qquad (\underset{\sim}{\boldsymbol{B}}, \boldsymbol{A}) \mapsto \left\langle \underset{\sim}{\boldsymbol{B}}, \boldsymbol{A} \right\rangle_{\mathscr{V}} \in \mathbb{R}, \tag{1.1}$$

where for vectors $\mathscr{F} = \mathbb{R}^3$. We use the same bra–ket notation $(\langle \bullet, \bullet \rangle_{\mathscr{V}})$ for the double contraction of compatible second order tensors defined on $\mathscr{F} = \mathbb{R}^3 \otimes \mathbb{R}^3$ and on $\mathscr{F} = \mathrm{sym}\{\mathbb{R}^3 \otimes \mathbb{R}^3\}$ respectively. Whenever there is no ambiguity to which

[1] $\tilde{\varrho}^{\mathrm{F}} = \tau^{\mathrm{F}} \lambda_{\mathrm{F}}$ where τ^{F} is the fibre tension in the Kirchhoff sense.

Please cite this article in press as: A. Zdunek, W. Rachowicz, A mixed finite element formulation for slightly compressible finite elasticity with stiff fibre reinforcement. Two fibre families. Uniaxial tension formulation, Computers and Mathematics with Applications (2018), https://doi.org/10.1016/j.camwa.2017.12.020.