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An efficient numerical algorithm for multi-dimensional time dependent partial differential equations

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ABSTRACT

An efficient and robust numerical scheme based on Haar wavelets and finite differences is suggested for the solution of two-dimensional time dependent linear and nonlinear partial differential equations (PDEs). Excellent feature of the scheme is the conversion of linear and non-linear PDEs to algebraic equations which are comparatively easy to handle. Convergence of the scheme, which guarantees small error norm as the resolution level increases, is also an important part of this work. Different error norms are computed to check efficiency of the technique. Computations verify accuracy, flexibility and low computational cost of the method.

1. Introduction

Various physical phenomenon such as heat conduction, acoustic waves and dynamic systems can be modeled as PDEs. Disparity in temperature with the passage time can be described in a region via heat equation. Diffusion equation is more generic form of heat equation arising in the study of chemical diffusion and relevant processes. Basic models of flow phenomena in transport problems involve the time dependent PDEs. In such studies Burgers' equation is extensively used in turbulence. Exact solution of nonlinear PDEs is not that simple to compute. Therefore, numerical techniques are the best choices for the solution of such problems. Among variety of numerical methods wavelets based approximation techniques turn out an effective tool for solving PDEs since 1990s [1,2]. For review of wavelets we refer to the papers [3–10]. To apply wavelet method one need to compute wavelet coefficients. For maximum wavelets this is somewhat tricky problem and needs special efforts. In all kinds of wavelets the simplest family is Haar wavelets, which is composed of rectangular box functions. Primarily the idea of these functions was presented by Alfred Haar in 1910. Haar functions are mathematically quite simple but are discontinuous at the partitioning points of the interval and hence not differentiable at these points. Due to this reason the direct implementation of Haar functions for differential equations is not possible.

To avoid this difficulty Cattani [11,12] used interpolating splines to regularize Haar wavelets. Alternate approach was used by Chen and Hasio [13]. They showed to approximate the highest order derivative with Haar wavelet series. Later on this technique was applied to solve variety of differential equations. Lepik [14,15] introduced a numerical method for the solution of PDEs using one and two dimensional Haar wavelets. Jiwari [16] used Haar wavelets coupled with quasi-linearization for solution of Burgers' equation. Mittal et al. [17] studied system of viscous Burgers' equations with the help of Haar wavelets. Oruc [18,19] applied finite difference hybrid scheme combined with Haar wavelets for the solution of modified Burgers' and KDV equations. Kumar [20] solved system of Burgers' equations by finite difference Haar wavelet technique. Somayeh et al. [21,22] investigated semi-analytical approach for solving Hunter–Saxton and foam drainage equations. The same authors [23] implemented Haar wavelets based scheme for the solution of two dimensional system of PDEs. Mittal and

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Pandit [32] solved unsteady squeezing nanofluid problems via Haar wavelets. In present article, we use Haar wavelets joined with finite difference for the solution of two dimensional time dependent linear and non-linear PDEs and system of PDEs.

The paper is planed as follows: Preliminaries of the Haar wavelets and their integrals are discussed in Section 2. Method description is given in Section 3. Convergence of the scheme is included in Section 4 while test problems are added in Section 5. In Section 6 conclusion of the paper is reported.

2. Haar wavelets and their integrals

To define Haar wavelets for arbitrary domain $[\xi, \eta]$, we refer to [15] for some basic results. Let us consider $2M = 2^{J+1}$, where J denotes maximal level of resolution. Also define parameters, $\lambda = 0, 1, \ldots, J;\ \vartheta = 0, 1, \ldots, \mu-1;\ \mu = 2^{\lambda}$, showing decomposition of wavelet number $i = \mu + \vartheta + 1$. The scaling functions for 1st and ith Haar wavelets are given by:

$$F_1(x) = \begin{cases} 1, & x \in [\xi, \eta] \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

$$F_i(x) = \begin{cases} (-1)^{j+1}, & x \in \left[\gamma_j(k), \gamma_{j+1}(k)\right), \quad j = 1, 2 \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

where

$$\gamma_{s+1}(k) = \xi + (2\vartheta + s)\,\alpha\delta x,\ \ s = 0, 1, 2 \text{ and } \alpha = M/\mu.$$

To solve nth order PDEs the following integrals are required

$$\varphi_{\kappa,i}(x) = \int_{\xi}^{x}\int_{\xi}^{x}\cdots\int_{\xi}^{x} F_i(z)dz^{\kappa} = \frac{1}{(\kappa-1)!}\int_{\xi}^{x}(x-z)^{\kappa-1}F_i(z)dz, \tag{2.3}$$

where

$$\kappa = 1, 2, \ldots n,\ \ i = 1, 2, \ldots 2M,\ \ \delta x = \frac{\xi - \eta}{2M}.$$

Keeping in view Eqs. (2.1) and (2.2), the analytical expressions of above integrals are given as follows:

$$\varphi_{\kappa,1}(x) = \frac{(x-\xi)^{\kappa}}{\kappa!}. \tag{2.4}$$

$$\varphi_{\kappa,i}(x) = \begin{cases} 0, & x < \gamma_1(x) \\ \frac{1}{\kappa!}[x-\gamma_1(i)]^{\kappa} & x \in [\gamma_1(i), \gamma_2(i)) \\ \frac{1}{\kappa!}\left[(x-\gamma_1(i))^{\kappa} - 2(x-\gamma_2(i))^{\kappa}\right] & x \in [\gamma_2(i), \gamma_3(i)) \\ \frac{1}{\kappa!}\left[(x-\gamma_1(i))^{\kappa} - 2(x-\gamma_2(i))^{\kappa} + (x-\gamma_3(i))^{\kappa}\right]. & x > \gamma_3(i). \end{cases} \tag{2.5}$$

3. Method description

In this section, the proposed scheme is discussed for linear PDEs. The same procedure can be extended to nonlinear and system of PDEs as well. For this purpose, let us consider two-dimensional heat equation

$$\partial_t w(x, y, t) = \partial_{xx} w(x, y, t) + \partial_{yy} w(x, y, t), \quad (x, y) \in \mho, \quad t > 0 \tag{3.1}$$

with Dirichlet boundary and initial conditions

$$w\,(x, y, t) = \chi(x, y, t), \quad (x, y) \in \partial\mho, \quad t > 0 \tag{3.2}$$

$$w\,(x, y, 0) = \psi(x, y), \quad (x, y) \in \mho \tag{3.3}$$

where $w(x, y, t)$ is targeted solution, $\mho \subseteq R^2$ domain, $\partial\mho$ boundary of $\mho$, and χ, ψ are known functions. Applying θ-weighted $(0 \le \theta \le 1)$ scheme to spatial part and forward difference to temporal part of Eq. (3.1) yields

$$w^{J+1} - \delta t\,\theta\left[\partial_{xx}w + \partial_{yy}w\right]^{J+1} = w^{J} + \delta t\,(1-\theta)\left[\partial_{xx}w + \partial_{yy}w\right]^{J} \tag{3.4}$$

where $w^J = w(x, y, t^J)$, $\quad t^{J+1} = \delta t + t^J$, and δt is time step. Now approximate mixed order derivative by two-dimensional Haar wavelets as follows:

$$\partial_{xxyy} w^{J+1}(x, y) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} \beta_{i,l}\, F_i(x)\, F_l(y) \tag{3.5}$$