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Blow-up phenomena of solutions for a reaction–diffusion equation with weighted exponential nonlinearity

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ABSTRACT

Blow-up phenomena for a reaction–diffusion equation with weighted exponential reaction term and null Dirichlet boundary condition are investigated. We establish sufficient conditions to guarantee existence of global solution or blow-up solution under appropriate measure sense by virtue of the method of super–sub solutions, the Bernoulli equation and the modified differential inequality techniques. Moreover, upper and lower bounds for the blow-up time are found in higher dimensional spaces and some examples for application are presented.

1. Introduction

We consider the initial and boundary value problem for a reaction–diffusion equation with weighted exponential reaction term

$$u_t = \Delta u + a(x)e^{\alpha u}, \quad (x,t) \in \Omega \times (0, t^*), \tag{1.1}$$

$$u = 0, \quad (x,t) \in \partial\Omega \times (0, t^*), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded region with smooth boundary $\partial\Omega$ and t^* represents the blow-up time when blow-up occurs or $t^* = +\infty$. The power α is a positive constant and the weight function $a \in C(\bar{\Omega})$ satisfies

$(\mathrm{a}_1):\ \ a(x) > 0, x \in \Omega$ and $a(x) = 0, x \in \partial\Omega$,

or

$(\mathrm{a}_2):\ \ a(x) \geq c > 0$ for all $x \in \bar{\Omega}$.

Moreover, the nonnegative initial data $u_0(x)$ is a continuous function which satisfies a compatibility condition. Therefore, by the standard parabolic theory, one can deduce that problem (1.1)–(1.3) has a unique nonnegative classical solution.

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The exponential reaction model (1.1) arises in many applications. For instance, model (1.1) is accounted for the solid fuel ignition model in the combustion theory, which can describe the non-dimensional ignition model for a supercritical high activation energy thermal explosion of a solid fuel in a bounded container Ω, refer to [1]. By virtue of a simple transformation, model (1.1) can be translated into a reaction–diffusion equation with a gradient term, which appears in many certain physical models. Such as the ballistic deposition processes, the evolution of the profile of a growing interface can be described by diffusive Hamilton–Jacobi type equations (cf. [2]). Meanwhile, there are many important phenomena formulated as the steady state equation for (1.1). For example, it describes problems of thermal self-ignition (cf. [3]), a ball of isothermal gas in gravitational equilibrium proposed by lord Kelvin (cf. [4]), the problem of temperature distribution in an object heated by the application of a uniform electric current (cf. [5]), and Osanger's vortex model for turbulent Euler flows (cf. [6]). It is also of interest in differential geometry (cf. [7]) and other applications.

In the past decades, many authors have been devoted to the investigation on existence and nonexistence of global solutions, blow-up phenomena, and asymptotic behavior of the solutions to reaction–diffusion equations. One can refer to monographs [1,8–11] as well as the survey paper [12,13] and the references therein. Specially, Bebernes and Eberly [1], Fila [9, Chapter 2], and Quittner and Souplet [10, Chapter 2] introduced the qualitative properties of the solution to the reaction–diffusion equation with constant coefficient, i.e., the weight function $a(x) = a > 0$, exponential nonlinearity and null Dirichlet boundary condition. Roughly, occurrence and type of blow-up depend on the constant a, the initial data, and the domain. Indeed, such a reaction term with exponential growth does not seem to be so much investigated than the polynomial growth reaction term for the reaction–diffusion equation. In this paper, we would like to investigate blow-up phenomena of the solution for a reaction–diffusion model with weighted exponential reaction term, and our main purpose is to derive bounds for the blow-up time if the blow-up occurs in finite time. As far as we know, a variety of methods have been used to study upper bounds for the blow-up time to these problems (cf. [14]). However, due to the explosive nature of the solutions, it is very important in applications to instead determine lower bounds for the blow-up time. Recently, some researches on lower bound for the blow-up time had been progressed, where most of articles are devoted to the investigation of polynomial growth reaction term for a reaction–diffusion model. One can find a review of previous results in [15–20] (three-dimensional space case) and [21] (higher dimensional spaces case) and references therein.

Concerning the research on the reaction–diffusion model with weighted polynomial growth reaction terms, one can refer to [22–24] and the references therein. Ma and Fang [22] studied the following semilinear reaction–diffusion equation with weighted inner source terms

$$u_t = \Delta u + a(x)f(u), \quad (x, t) \in \Omega \times (0, t^*), \tag{1.4}$$

under Robin boundary condition, where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary, the weight function $a(x)$ satisfies (a_1) or (a_2), and the nonlinear nonnegative function $f(u)$ satisfies appropriate nonlocal conditions. Under some suitable conditions, they established sufficient conditions for solutions of global existence and blowing up at finite time, and derived bounds for the blow-up time of the solution in higher dimensional spaces under appropriate measure sense. Meanwhile, they [23,24] considered the reaction model with local or nonlocal weighted inner absorption and nonlinear boundary flux.

Recently, in addition to monographs [1,9,10] on the exponential reaction model, Pulkkinen [25] considered the stability and blow-up of the solution for model (1.1) with $a(x) = 1$ and $\alpha = 1$. Tello [26] discussed large time behavior for solutions of the Cauchy problem. Ioku [27] investigated the Cauchy problem for a heat equation with another exponential nonlinearity e^{u^2} and then proved the existence of global solutions. Afterwards, Dai and Zhang [28] considered the energy decay result and nonexistence of global solution for an initial and Dirichlet boundary problem of a reaction–diffusion equation with generalized Lewis function and nonlinear exponential growth by potential well theory.

To our knowledge, the study on blow-up analysis for the reaction–diffusion model with weighted exponential nonlinearity (1.1)–(1.3) has not been uncovered yet in the higher dimensional spaces $(N \geq 1)$. At a glance, the main difficulty lies in finding the influence of weight function $a(x)$ and exponential nonlinearity to the blow-up phenomena. We pay our attention to establish sufficient conditions to guarantee existence of global solution to problem (1.1)–(1.3) or blow-up at finite time under appropriate measure sense. Moreover, upper and lower bounds for the blow-up time are derived in higher dimensional spaces.

The rest of our paper is organized as follows: In Section 2, we construct super–sub solutions of problem (1.1)–(1.3) to get existence and nonexistence of global solution. In Section 3, we use the Bernoulli equation technique to prove the sufficient condition to ensure blow-up of the solution to problem (1.1)–(1.3) at finite time and obtain an upper bound for the blow-up time. Moreover, we introduce a condition that the solution blows up, at not a finite time. In Section 4, we are devoted to drive lower bounds for the blow-up time under three different measures in the higher dimensional spaces. Finally, a few examples are given to illustrate applications of our main results in Section 5.

2. Existence and nonexistence of global solution

In this section, we would utilize the method of super–sub solutions, which is coupled with Bernoulli's equation technique to obtain the well-known previous result. That is, problem (1.1)–(1.3) has a global solution for sufficiently small initial data; while the problem has no global solution for large initial data.

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