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Trigonometric transform splitting methods for real symmetric Toeplitz systems

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ABSTRACT

In this paper we study efficient iterative methods for real symmetric Toeplitz systems based on the trigonometric transformation splitting (TTS) of the real symmetric Toeplitz matrix A. Theoretical analyses show that if the generating function f of the $n \times n$ Toeplitz matrix A is a real positive even function, then the TTS iterative methods converge to the unique solution of the linear system of equations for sufficient large n. Moreover, we derive an upper bound of the contraction factor of the TTS iteration which is dependent solely on the spectra of the two TTS matrices involved.

Different from the CSCS iterative method in Ng (2003) in which all operations counts concern complex operations when the DFTs are employed, even if the Toeplitz matrix A is real and symmetric, our method only involves real arithmetics when the DCTs and DSTs are used. The numerical experiments show that our method works better than CSCS iterative method and much better than the positive definite and skew-symmetric splitting (PSS) iterative method in Bai et al. (2005) and the symmetric Gauss–Seidel (SGS) iterative method.

1. Introduction

Consider the iterative solution to the following linear system of equations

$$Ax = b \tag{1.1}$$

by the two-step splitting iteration with alternation, where $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite Toeplitz matrix.

Very often, a Toeplitz matrix A is generated by a function $f(x) \in \mathbf{C}_{2\pi}$, i.e., $[A]_{m,k} = a_{m-k}$, where

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta\ ,\ k = 0, \pm 1, \pm 2, \ldots \tag{1.2}$$

the function f is called the generating function of the Toeplitz matrix A.

Toeplitz linear systems arise in a variety of applications in mathematics, scientific computing and engineering, for instance, image restoration storage problems in signal processing, algebraic differential equation, time series and control theory. Those applications have motivated both mathematicians and engineers to develop specific algorithms catering to solving Toeplitz systems for instance, [1–3] and references therein.

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Iterative methods for the linear system of equations (1.1) require efficient splittings of the coefficient matrix A. For example, the Jacobi and the Gauss–Seidel iterations [3,4] split the matrix A into its diagonal part D, and strictly lower triangular part L and upper triangular part U, respectively, and the generalized conjugate gradient (CG) method and the generalized Lanczos method splits the matrix A into its Hermitian and skew-Hermitian parts, see for example [5] and references therein.

Based on the fact that every matrix A naturally has a Hermitian/skew-Hermitian splitting (HSS)

$$A = H + S \quad \text{with} \quad H = \frac{A + A^*}{2} \quad \text{and} \quad S = \frac{A - A^*}{2}. \tag{1.3}$$

Bai et al. presented the following HSS iteration in [5] to solve the non-Hermitian positive definite linear system of equations.

The HSS iteration. *Given an initial guess $x^{(0)}$. For $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute*

$$\begin{cases}(\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b\\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b,\end{cases} \tag{1.4}$$

where α is a given positive constant.

Evidently, each iterate of the HSS iteration alternates between the Hermitian part H and the skew-Hermitian part S of the matrix A, analogously to the classical alternating direction implicit iteration (ADI) introduced by Peaceman and Rachford for solving partial differential equations, see [4,6]. Results associated to the stationary iterative method with alternation can be also found in Benzi and Szyld [7]. Theoretical analysis shows that the HSS iteration (1.4) converges unconditionally to the unique solution of the system of linear equations (1.1), if the coefficient matrix A is a non-Hermitian positive definite matrix. Due to its promising performance and elegant mathematical properties, the HSS scheme immediately attracted considerable attention, resulting in numerous papers devoted to various aspects of this new method, see for instance [8–17] and references therein.

Recently, Gu and Tian in [16] and Chen and Jiang in [15] have respectively tailored the HSS iteration for real positive definite Toeplitz systems by using the reducible properties of symmetric and skew symmetric Toeplitz matrices. That is that each of two systems in (1.4) is reduced to two subsystems with about half sizes and the computational complexity of their methods is reduced to about half of the HSS iteration at each iteration. However, the computational costs remain $O(n^2)$ flops at each iteration. This seems to be not a very good motivation, because every $n \times n$ Toeplitz matrix A enjoys a circulant and skew-circulant splitting $A = C + S$ (denoted by CSCS) which results in the following CSCS iteration proposed by Ng in [18] for non-Hermitian positive definite Toeplitz systems.

The CSCS iteration. *Given an initial guess $x^{(0)}$. For $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute*

$$\begin{cases}(\alpha I + C)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b\\ (\alpha I + S)x^{(k+1)} = (\alpha I - C)x^{(k+\frac{1}{2})} + b,\end{cases} \tag{1.5}$$

where α is a positive constant.

Evidently, the matrices $\alpha I \pm C$ and $\alpha I \pm S$ are circulant matrices and skew-circulant matrices, respectively, which can be diagonalized by the discrete Fourier matrix F and $\hat{F}$, where $\hat{F} = F\Omega$ with Ω a diagonal matrix; see for example [2,3,19]. Thus, the cost of each iteration in (1.5) is $O(n \log n)$ complex flops by using 8 FFTs of n-vectors.

It is shown in [18] that if the real part of the generating function f is positive, then the generated Toeplitz matrix A is positive definite. Furthermore, the splitting matrices C and S are also positive definite for large enough n. In this case the CSCS iteration converges unconditionally to the unique solution of the system of linear equations (1.1). In particular, if the generating function f is real positive even, then the generated Toeplitz matrix A is real symmetric positive definite. Therefore, we may use the CSCS iteration for solving the system of linear equations (1.1) by using FFTs. However, all operation counts above concern complex operations. It is desirable to develop an analogue of (1.5) only involving real arithmetics. This is the main motivation of this paper.

This paper is organized as follows. In Section 2 we first review some basic definitions, notation and preliminaries related to the splittings of the real symmetric Toeplitz matrices based on trigonometric transformations. Then we develop an iterative method based on the splitting above, which is referred to as trigonometric transform splitting and briefly denoted by TTS, for solving (1.1). We finally study the convergence properties and analyze the convergence rate of the TTS iteration. Some computational details are also discussed. Numerical experiments are presented in Section 3 to show the effectiveness of our methods. A brief conclusion is also drawn.

2. The TTS iteration

In this section we first review the trigonometric transform based splitting of a real symmetric Toeplitz matrix, then discuss the convergence of the TTS iteration, and finally give the details of implementation of the TTS iteration as well as computational complexity.

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