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Some preconditioners for elliptic PDE-constrained optimization problems[☆]

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ABSTRACT

For the structured systems of linear equations arising from the Galerkin finite element discretizations of elliptic PDE-constrained optimization problems, some preconditioners are proposed to accelerate the convergence rate of Krylov subspace methods such as GMRES for both cases of the Tikhonov parameter β not very small (equal or greater than 1e−6) and sufficiently small (less than 1e−6), respectively. We derive the explicit expressions for the eigenvalues and eigenvectors of the corresponding preconditioned matrices. Numerical results show that the corresponding preconditioned GMRES methods perform and match well with the theoretical results.

1. Introduction

Consider the following linear elliptic distributed optimal control problems:

$$\min_{u,f} \ \frac{1}{2}\|u-u_*\|^2_{L^2(\Omega)}+\beta\|f\|^2_{L^2(\Omega)}, \tag{1.1}$$

$$\text{s.t.} \ -\Delta u = f \quad \text{in } \Omega, \tag{1.2}$$

$$u = g \quad \text{on } \partial\Omega_1 \quad \text{and} \quad \frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega_2, \tag{1.3}$$

where Ω is a simply connected polygonal domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a connected boundary $\partial\Omega$, $\partial\Omega_1$ and $\partial\Omega_2$ are two parts of $\partial\Omega$ satisfying $\partial\Omega_1\cup\partial\Omega_2=\partial\Omega$ and $\partial\Omega_1\cap\partial\Omega_2=\emptyset$. Such problems, introduced by Lions in [1], consist of a cost function to be minimized subject to a partial differential equation (PDE) problem posed on the domain Ω. Here, the function u_* is the desired state, β is the Tikhonov regularization parameter, u is the state variable, and f is called the control variable.

Recently, based on discretize-then-optimize approach, Rees et al. [2] transformed the PDE-constrained optimization problem (1.1)–(1.3) into a linear system. More concretely, by using the Galerkin finite element method to the weak

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formulation of (1.2) and (1.3), we obtain the following corresponding minimization problem

$$\min_{\mathbf{u},\mathbf{f}} \ \frac{1}{2}\mathbf{u}^{\mathrm{T}}M\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{b} + \|\mathbf{u}_*\|_2^2 + \beta\mathbf{f}^{\mathrm{T}}M\mathbf{f},$$
$$\text{s.t. } K\mathbf{u} = M\mathbf{f} + \mathbf{d}, \tag{1.4}$$

where $M \in \mathbb{R}^{m\times m}$ is the mass matrix, $K \in \mathbb{R}^{m\times m}$ is the stiffness matrix (the discrete Laplacian), $\mathbf{d} \in \mathbb{R}^m$ contains the terms coming from the boundary values of the discrete solution, and $\mathbf{b} \in \mathbb{R}^m$ is the Galerkin projection of the discrete state u_*. Then, by applying the Lagrange multiplier technique to the minimization problem (1.4), we can obtain the following linear system

$$\mathcal{A}\mathbf{x} \equiv \begin{pmatrix} 2\beta M & 0 & -M \\ 0 & M & K^{\mathrm{T}} \\ -M & K & 0 \end{pmatrix}\begin{pmatrix} \mathbf{f} \\ \mathbf{u} \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{b} \\ \mathbf{d} \end{pmatrix} \equiv \mathbf{g}, \tag{1.5}$$

where $\lambda \in \mathbb{R}^m$ is a vector of Lagrange multipliers; see [3].

There have existed several preconditioners for the linear system (1.5). For example, in [2] the MINRES method coupled with the block-diagonal preconditioner

$$\mathcal{P}_{\mathrm{D}} = \begin{pmatrix} 2\beta M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & KM^{-1}K^{\mathrm{T}} \end{pmatrix}$$

and the projected preconditioned conjugate gradient (PPCG) method incorporated with the constraint preconditioner

$$\mathcal{P}_{\mathrm{C}} = \begin{pmatrix} 0 & 0 & -M \\ 0 & 2\beta K^{\mathrm{T}}M^{-1}KM & K^{\mathrm{T}} \\ -M & K & 0 \end{pmatrix};$$

in [4], the GMRES methods coupled with the block-counter-diagonal preconditioner

$$\mathcal{P}_{\mathrm{BCD}} = \begin{pmatrix} 0 & 0 & -M \\ 0 & M & 0 \\ -M & 0 & 0 \end{pmatrix}$$

and the block-counter-tridiagonal preconditioner

$$\mathcal{P}_{\mathrm{BCT}} = \begin{pmatrix} 0 & 0 & -M \\ 0 & M & K^{\mathrm{T}} \\ -M & K & 0 \end{pmatrix};$$

in [5], the GMRES methods coupled with the block-symmetric preconditioner

$$\mathcal{P}_{\mathrm{BS}} = \begin{pmatrix} 2\beta M & 0 & -M \\ 0 & M & 0 \\ -M & 0 & 0 \end{pmatrix}$$

and the block-lower-triangular preconditioner

$$\mathcal{P}_{\mathrm{BLT}} = \begin{pmatrix} 2\beta M & 0 & 0 \\ 0 & M & 0 \\ -M & K & -\frac{1}{2\beta}M \end{pmatrix};$$

in [6], Rees and Stoll proposed the following 'ideal' preconditioner

$$\mathcal{P}_{\mathrm{ideal}} = \begin{pmatrix} 2\beta M & 0 & 0 \\ 0 & M & 0 \\ -M & K & KM^{-1}K^{\mathrm{T}} \end{pmatrix};$$

in [7], Pearson and Wathen presented Schur complement approximation with block diagonal and block triangular preconditioners for PDE constrained optimization as follows

$$\widehat{\mathcal{P}}_1 = \begin{pmatrix} 2\beta\widetilde{M} & 0 & 0 \\ 0 & \widetilde{M} & 0 \\ 0 & 0 & \widetilde{S} \end{pmatrix} \quad \text{and} \quad \widehat{\mathcal{P}}_2 = \begin{pmatrix} 2\gamma\beta\widetilde{M} & 0 & 0 \\ 0 & \gamma\widetilde{M} & 0 \\ -M & K & \widetilde{S} \end{pmatrix},$$