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A two-level linearized compact ADI scheme for two-dimensional nonlinear reaction–diffusion equations[☆]

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ABSTRACT

A novel two-level linearized compact alternating direction implicit (ADI) scheme is proposed for solving two-dimensional nonlinear reaction–diffusion equations. The computational cost is reduced by use of the Newton linearized method and the ADI method. The existence and uniqueness of the numerical solutions are proved. Moreover, the error estimates in H^1 and L^∞ norms are presented. Numerical examples are given to illustrate our theoretical results.

1. Introduction

We are interested in developing a two-level linearized compact ADI scheme for solving the following two-dimensional reaction–diffusion equation

$$\frac{\partial u}{\partial t} - a\Delta u = f(u), \qquad (x, y) \in \Omega, \ \ t \in (0, T], \tag{1.1}$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega, \tag{1.2}$$

$$u(x, y, t) = \varphi(x, y, t), \quad (x, y) \in \partial\Omega, \ \ t \in (0, T], \tag{1.3}$$

where a is a positive constant, $f \in C^2(\mathbb{R})$, Δ denotes the Laplacian operator, $\Omega = (0, 1) \times (0, 1)$ and $\partial\Omega$ is the boundary. In this study, the initial condition u_0 and the boundary condition φ are assumed to be regular enough and satisfy the condition that $\varphi(x, y, 0) = u_0(x, y)$ when $(x, y) \in \partial\Omega$ so that the initial–boundary value problem (1.1)–(1.3) admits a smooth solution.

The simplest time discretization for solving the nonlinear reaction–diffusion equation (1.1)–(1.3) is the explicit numerical schemes. However, they usually suffer the severely restricted temporal step-size [1,2]. Another strategy is to apply the fully implicit schemes. Although the schemes are usually unconditionally stable, one has to solve a large system of nonlinear equations at every time level [3–6]. It leads to a considerable computational cost in practical application. A possible

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improvement is to use the linearized implicit (or semi-implicit) schemes. As far as we know, the most widely used second-order time discretization is the linearized Crank–Nicolson method [7–12] or q-step implicit–explicit schemes [13–19]. These schemes require at least two starting values, which are obtained by the initial values and an additional predictor schemes or iterative schemes.

The goal of this paper is to develop a linearized compact ADI scheme for solving the nonlinear reaction–diffusion equations (1.1)–(1.3) and present optimal error estimates in the sense of H^1 and L^∞ norms. The time discretization is achieved via a combination of Crank–Nicolson scheme and a Newton linearized method for the nonlinear term, and the spatial discretization is performed by using the compact finite difference method. The highlights of the present work are

- The proposed method is a linearized and two-level scheme, which only requires only one starting value. This is sharp contract to the Crank–Nicolson method or q-step implicit–explicit schemes.
- By applying the ADI method, the two-dimensional reaction–diffusion equation is reduced to independent systems of one-dimensional problems. The computational cost is reduced.
- For the nonlinear problems, it is proved that the proposed scheme has the convergence order of 2 in temporal direction and the convergence order of 4 in spatial direction in H^1 and L^∞ norms, respectively, while the previous investigations mainly focus on error estimates for the linear problems [20–22] or L^2-error estimates for nonlinear problems [10,23,24].

The rest of the paper is organized as follows. In Section 2, we propose the Newton linearized compact ADI scheme for solving the problem (1.1)–(1.3). In Section 3, the convergence of the scheme is investigated. In Section 4, we present several numerical experiments to verify the theoretical results. Finally, we give some conclusions in Section 5.

2. Construction of the two-level linearized compact ADI scheme

In this section, we present the two-level linearized compact ADI scheme for solving problem (1.1)–(1.3).

Let $\tau = \frac{T}{K}$ and $h = \frac{1}{M}$ be the temporal and spatial stepsize, respectively, where K and M are two positive integers. Set $t_k = k\tau$, $t_{k-\frac{1}{2}} = \frac{t_k + t_{k-1}}{2}$ $(k = 1, \ldots, K)$, $\Omega_\tau = \{t_k | 0 \le k \le K\}$, $x_i = ih, y_j = jh$, $(0 \le i, j \le M)$ and $\bar{\Omega}_h = \{(x_i, y_j) | 0 \le i, j \le M\}$, $\Omega_h = \bar{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega$. Let $\mathcal{V}_h = \{v_{i,j}^k | \, i, j = 0, 1, 2, \ldots, M, k = 0, 1, \ldots, K\}$ be grid function space defined on $\Omega_h \times \Omega_\tau$. Define

$$v_{i,j}^{k+\frac{1}{2}} = \frac{v_{i,j}^{k+1} + v_{i,j}^k}{2},\ \delta_t v_{i,j}^{k+\frac{1}{2}} = \frac{v_{i,j}^{k+1} - v_{i,j}^k}{\tau},\ \delta_x v_{i+\frac{1}{2},j}^k = \frac{v_{i+1,j}^k - v_{i,j}^k}{h},\ \delta_y v_{i,j+\frac{1}{2}}^k = \frac{v_{i,j+1}^k - v_{i,j}^k}{h},$$

$$\delta_x^2 v_{i,j}^k = \frac{\delta_x v_{i+\frac{1}{2},j}^k - \delta_x v_{i-\frac{1}{2},j}^k}{h},\ \delta_y^2 v_{i,j}^k = \frac{\delta_y v_{i,j+\frac{1}{2}}^k - \delta_y v_{i,j-\frac{1}{2}}^k}{h},\ \Delta_h v_{i,j}^k = \delta_x^2 v_{i,j}^k + \delta_y^2 v_{i,j}^k.$$

Denote

$$\|v^k\|_\infty = \max_{1 \le i,j \le M-1} |v_{i,j}^k|, \quad \|v^k\| = h\sqrt{\sum_{i,j=1}^{M-1} |v_{i,j}^k|^2}, \quad \|\delta_x v^k\| = h\sqrt{\sum_{i=0}^{M-1}\sum_{j=1}^{M-1} |\delta_x v_{i+\frac{1}{2},j}^k|^2},$$

$$\|\delta_x^2 v^k\| = h\sqrt{\sum_{i,j=1}^{M-1} |\delta_x^2 v_{i,j}^k|^2},\ \|\delta_x\delta_y v^k\| = h\sqrt{\sum_{i,j=0}^{M-1} |\delta_x\delta_y v_{i+\frac{1}{2},j+\frac{1}{2}}^k|^2},\ \|\Delta_h v^k\| = h\sqrt{\sum_{i,j=1}^{M-1} |\Delta_h v_{i,j}^k|^2},$$

similarly, norms $\|\delta_y v^k\|$, $\|\delta_y^2 v^k\|$, $\|\delta_y\delta_x v^k\|$ can also be defined. The H^1 semi-norm and H^1 norm are defined as $|v^k|_1 = \sqrt{\|\delta_x v^k\|^2 + \|\delta_y v^k\|^2}$ and $\|v^k\|_1 = \sqrt{\|v^k\|^2 + |v^k|_1^2}$, respectively.

Besides, we introduce the compact difference operators

$$\mathcal{A}v_{i,j}^k = \begin{cases} (1 + \dfrac{h^2}{12}\delta_x^2)v_{i,j}^k, & 1 \le i \le M-1,\ 0 \le j \le M, \\ v_{i,j}^k & i = 0 \text{ or } M,\ 0 \le j \le M, \end{cases}$$

$$\mathcal{B}v_{i,j}^k = \begin{cases} (1 + \dfrac{h^2}{12}\delta_y^2)v_{i,j}^k, & 0 \le i \le M,\ 1 \le j \le M-1, \\ v_{i,j}^k & 0 \le i \le M,\ j = 0 \text{ or } M. \end{cases}$$

The following lemma was proved in [25]. It plays an important role in developing the compact scheme.

Lemma 1 ([25]). *Assume that* $g(x) \in C^6[x_{i-1}, x_{i+1}]$. *Then*

$$\frac{1}{12}\left[g''(x_{i-1}) + 10g''(x_i) + g''(x_{i+1})\right] - \frac{1}{h^2}\left[g(x_{i-1}) - 2g(x_i) + g(x_{i+1})\right] = \frac{h^4}{240}g^{(6)}(\omega_i),$$

where $\omega_i \in (x_{i-1}, x_{i+1})$.

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