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# The new exact solitary and multi-soliton solutions for the (2+1)-dimensional Zakharov–Kuznetsov equation

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ABSTRACT

In this paper, by employing two different simplest equation methods, the (2+1)-dimensional Zakharov–Kuznetsov (ZK) equation derived for describing weakly nonlinear ion-acoustic waves in the plasma is investigated. With the aid of the Bernoulli equation and the coupled Burgers' equations, the electric field potential of ZK equation are formally obtained, which are presented as the new solitary and multi-soliton solutions. Meanwhile, the electric field and magnetic field can be accordingly obtained. In addition, the significant features of the variable coefficient and parameter are discovered. The results show that the solitary and multi-soliton solutions are precisely obtained and the efficiency of the methods is demonstrated. These new exact solutions will extend previous results and help to explain the features of nonlinear ion-acoustic waves in the plasma.



## 1. Introduction

In the past decades the nonlinear physical phenomena are related to nonlinear evolution equations (NLEEs), which are involved in many fields. As mathematical models of the phenomena, the investigation of exact solutions of NLEEs will help one to understand these phenomena better. Therefore, the most important aim is to focus and carry out exact solitary solutions in a detailed manner from a mathematical point of view. In recent years, many powerful methods to construct solitary solutions of NLEEs have been established and developed, for example the Exp-function method [1–3], the tanh–coth method [4,5], the sine–cosine method [6], the first integral method [7,8], the $G'/G$-expansion method [9,10], and the simplest equation method (SEM) [11,12].

In this paper, the (2+1)-dimensional Zakharov–Kuznetsov (ZK), a generalization of the KdV equation, that governs the behavior of weakly nonlinear ion-acoustic waves in a plasma is examined [3–6,13–19]. To study well the ZK equation, it is necessary to present more much exact solutions. Therefore, the investigation of ion-acoustic waves and structures in plasma has attracted lots of attention. For example, Wazwaz [4–6,13–15,17] solved the ZK equation by the tanh–coth method [4,5] and sine–cosine method [6], and obtained eleven and four exact solutions, respectively. Aslan [3] applied the Exp-function method and obtained thirty three exact solutions. Seadawy [18] applied the extended direct algebraic method and presented four solitary solutions. And after employing the auxiliary equation method and Hirota bilinear method to handle the (3+1)-dimensional ZK equation, Zhang et al. [19] gave eleven traveling wave solutions, five solitary solutions, and 2 type multi-soliton solutions. According to the derived exact solutions, we can study the behavior of the wave propagation in plasma and find more interesting linear and nonlinear structures. For this purpose, we try to seek more new solitary and multi-soliton solutions. In this work, the simplest equation method with 2 different assumed simplest equations [20–27] is used to achieve our goal.

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The (2+1)-dimensional ZK equation illustrated in detail in [18,19] is given as

$$\phi_\tau + \mu\phi\phi_\zeta + \frac{1}{2}\phi_{\zeta\zeta\zeta} + \frac{1}{2}(1+\delta)\phi_{\eta\eta\zeta} = 0, \tag{1.1}$$

where $\mu$ is an arbitrary constant and $\delta$ is a parameter of electron temperature. Substituting the traveling wave transformation $k\zeta + m\eta - \omega\tau = \xi, \phi(\zeta, \eta, \tau) = \phi(\xi)$ into Eq. (1.1) and integrating once with respect to $\xi$ yields

$$-\omega\phi + \frac{\mu k}{2}\phi^2 + (\frac{k^3}{2} + \frac{km^2}{2}(1+\delta))\phi_{\xi\xi} = 0. \tag{1.2}$$

After applying the SEM to handle Eq. (1.2) two new solitary solutions and one generalized multi-soliton solution are formally obtained, respectively. In the following the algorithm will be illustrated in detail.

## 2. The simplest equation methods

In this section, the SEM with the aid of the Bernoulli equation and the coupled Burgers' equations as the simplest equation is presented, respectively [20–28].

First, considering a partial differential equation (PDE) and letting by means of an appropriate ansatz $\xi = k\zeta + m\eta - \omega\tau$ ($k$ and $m$ are wave numbers and $\omega$ is wave speed), the equation could be reduced to a nonlinear ordinary differential equation (ODE)

$$P\left(F(\xi), \frac{dF}{d\xi}, \frac{d^2F}{d\xi^2}, \ldots\right) = 0. \tag{2.1}$$

For the large class of equations from the kind (2.1) the exact solution can be constructed as finite series

$$F(\xi) = \sum_{i=0}^{L} a_i[Y(\xi)]^i, \tag{2.2}$$

where $a_i$ are constants to be determined, and $Y$ is the exact solution of some ODEs referred to as the simplest equations. $L$ is a constant to be determined which is the power of the specified solution function finite series, $Y$. The simplest equation is of lesser order than Eq. (2.1) and we know the general solution of the simplest equation [20–26].

The application steps of the SEM had been summarized as the following [21,25].

(i) Using the traveling wave ansatz the solved class of nonlinear partial differential equation (NPDE) is reduced to a class of nonlinear ODEs of like Eq. (2.1).
(ii) By means of a balance equation between the highest order derivative term and the highest order nonlinear term appearing in Eq. (2.1), the value of $L$ is given.
(iii) The finite-series solution (2.2) is substituted in (2.1) and as a result a polynomial of $Y$ is obtained.
(iv) Equating all the coefficients of $Y$ to zero yields a system which can be solved to find $a_i$. Substituting the values of $a_i$ into Eq. (2.2) completes the determination of the solution for Eq. (2.1).

So far, the SEM is illustrated. The most important step of applying the SEM is choosing the simplest equation whose exact solution is considered as the seed used to constructed new solutions for investigated nonlinear equations. In what follows, in order to seek new solitary solutions and multi-soliton solutions, the Bernoulli equation [21,24] and the coupled Burgers' equations [28] are set as the simplest equations, respectively.

**Model 1. The Bernoulli equation as the simplest equation**

The Bernoulli equation is a well-known ODE and considered as [21,24]

$$Y_\xi = aY + bY^2, \tag{2.3}$$

where $a$ and $b$ are arbitrary constants. Eq. (2.3) has the exact solution in the form of

$$Y = \frac{ae^{a(\xi)}}{1 - be^{a(\xi)}}. \tag{2.4}$$

**Model 2. The coupled Burgers' equations as the simplest equation**

In order to seek multi-soliton solutions of the ZK equation, the coupled Burgers' equations are chosen as the simplest equation due to they are the (2+1)-dimensional completely integrable equations [28]. These equations are given as

$$u_\tau - 2uu_\zeta - v_{\zeta\zeta} = 0, \tag{2.5}$$

$$v_{\eta\tau} - u_{\zeta\zeta\eta} - 2uv_{\zeta\eta} - 2u_\zeta v_\eta = 0. \tag{2.6}$$

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