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Numerical approximation of a time-fractional Black–Scholes equation

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ABSTRACT

In this paper a time-fractional Black–Scholes equation is examined. We transform the initial value problem into an equivalent integral–differential equation with a weakly singular kernel and use an integral discretization scheme on an adapted mesh for the time discretization. A rigorous analysis about the convergence of the time discretization scheme is given by taking account of the possibly singular behavior of the exact solution and first-order convergence with respect to the time variable is proved. For overcoming the possibly nonphysical oscillation in the computed solution caused by the degeneracy of the Black–Scholes differential operator, we employ a central difference scheme on a piecewise uniform mesh for the spatial discretization. It is proved that the scheme is stable and second-order convergent with respect to the spatial variable. Numerical experiments support these theoretical results.

1. Introduction

The valuation of option contracts has attracted much attention from both financial engineers and mathematicians during the last decades. A popular approach for pricing option contracts is the use of mathematics models based on partial differential equations. Black and Scholes [1] showed that the value of a European option on a stock, whose price follows a geometric Brownian motion with constant drift and volatility, is governed by a second-order parabolic partial differential equation with respect to time and stock price, which is known as the Black–Scholes (B–S) equation. However, it is well known that the hypotheses of the classical B–S equation are so idealistic that it is not completely consistent with the actual stock movement.

With discovery of the fractional structure of financial market, the fractional B–S equations are introduced to extend the financial theory. Wyss [2] first presented a time-fractional B–S equation to price a European call option. Jumarie [3,4] applied the fractional Taylor formula to derive the time and space fractional B–S equations. Cartea and del-Castillo-Negrete [5] proposed the fractional diffusion models for pricing exotic options in markets with jumps. Liang et al. [6,7] proposed bi-fractional Black–Scholes–Merton models of option pricing. Chen et al. [8] derived a time-fractional B–S equation which is a slightly simplified version of Liang et al.'s model [6,7].

In this paper we consider the following time-fractional B–S equation derived by Chen et al. [8]:

$$D_\tau^\alpha v + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv = 0, \quad (x, \tau) \in \mathbb{R}^+ \times [0, T), \tag{1.1}$$

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equipped with the terminal and boundary conditions:

$$v(x, T) = \max(x - E, 0), \qquad x \in \mathbb{R}^+, \tag{1.2}$$

$$v(0, \tau) = 0, \qquad \tau \in [0, T], \tag{1.3}$$

where x is the asset price, E is the exercise price, T is the maturity, r is the risk-free interest rate, σ is the volatility of underlying asset, $D_\tau^\alpha v$ is a right Riemann–Liouville fractional derivative defined as

$$D_\tau^\alpha v = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial \tau} \int_\tau^T \frac{v(x,s) - v(x,T)}{(s-\tau)^\alpha} ds, \qquad 0 < \alpha < 1. \tag{1.4}$$

For $\alpha = 1$, (1.1) reduces to the classical B–S equation [8].

Various numerical methods have been developed in recent years for the fractional B–S equations. Song and Wang [9] and Zhang et al. [10] employed implicit finite difference methods to solve the time-fractional B–S equations. Koleva and Vulkov [11] derived a weighted finite difference scheme for a time-fractional B–S equation. De Staelen and Hendy [12] gave a stable scheme with higher orders of convergence for pricing double barrier options in a time-fractional Black–Scholes model. Cartea and del-Castillo-Negrete [5] applied the shifted Grüwald–Letnik scheme to solve the FMLS model governing the valuation of European options. Chen and Wang [13] proposed a finite difference method with a power penalty method to solve a space-fractional parabolic variational inequality for pricing American options. Marom and Momoniat [14] presented a comparison of numerical solutions of the FMLS, KoBoL and CGMY models for pricing European options and obtained the conditions for the convergence of these models. Chen et al. [15] introduced a robust numerical method based on the upwind scheme for pricing American options under the generalized mixed fractional Brownian motion model. Chen et al. [16] also proposed a numerical evaluation technique for pricing European options under the CGMY model. Yang et al. [17] constructed a universal difference method to solve a time–space fractional B–S equation. But these papers deal only with very special cases by assuming (explicitly or implicitly) that the solutions of the fractional B–S equations are smooth on the closure of the domain where the problem is posed.

The exact solutions of the fractional differential equations may exist singularity near some domain boundaries, see [18–23] and references therein. Hence, there exists two difficulties for solving the time-fractional B–S equation (1.1)–(1.3). The first one is the failure of the standard numerical methods for giving accurate approximation of the fractional derivative due to the possibly singularity of the exact solution. The other one is the possibly nonphysical oscillation in the computed solution caused by the degeneracy of the B–S differential operator at $x = 0$. As far as we know, there is no paper which has taken into account the possibly singularity of the exact solutions for solving fractional B–S equations numerically. For dealing with the degeneracy of the B–S differential operator, a common and widely used approach is the Euler transformation, which may lead to cause computational errors and the originally grid points concentrating around $x = 0$ inappropriately [24,25]. Hence, the B–S equation in the original form needs to be solved.

In this paper, to deal with the fractional derivative with the possible singularity of the exact solution, the initial value problem is transformed into an equivalent integral–differential equation with a weakly singular kernel. Then an integral discretization scheme on an adapted mesh is proposed for the time discretization. A rigorous analysis about the convergence of the time discretization scheme is given by taking account of the possibly singularity of the exact solution. It is proved that the scheme is first-order convergent respect to the time variable. For overcoming the possibly nonphysical oscillation in the computed solution caused by the degeneracy of the B–S differential operator at $x = 0$, a central difference scheme on a piecewise uniform mesh is employed for the spatial discretization. It is proved that the scheme is stable and second-order convergent with respect to the spatial variable. Numerical experiments will display that this method is more accurate and robust than finite difference methods for approximating the fractional derivative when α is close to 0.

The remainder of the paper is organized as follows. In the next section we describe some theoretical results on the continuous time-fractional B–S equation. The time semidiscretization scheme is introduced in Section 3. The spatial discretization scheme is developed in Section 4. The fully discrete scheme is presented in Section 5. Numerical experiments are provided in Section 6.

Notation. Throughout the paper, C will denote a generic positive constant that is independent of the mesh. Note that C can take different values in different places. We always use the (pointwise) maximum norm $\|\cdot\|_Q$, where Q is a closed and bounded set.

2. The continuous problem

To facilitate the solution process, we first use the variable transformations: $t = T - \tau$ and $u(x,t) = v(x,\tau)$, from (1.4) to get

$$D_t^\alpha u = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{u(x,s) - u(x,0)}{(t-s)^\alpha} ds, \qquad 0 < \alpha < 1.$$

According to the relationship between the Riemann–Liouville fractional derivative and the Caputo derivative, Chen et al. [8, Lemma 3.1] showed

$$D_t^\alpha u = -\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial t}(x,s) ds \equiv -\frac{\partial^\alpha u}{\partial t^\alpha}, \qquad 0 < \alpha < 1.$$