Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Dichotomy of solutions to discrete p-Laplace equations and p-Laplace parabolic equations

Soon-Yeong Chung

Department of Mathematics and Department of Biomedical Engineering, Sogang University, Seoul 04107, Republic of Korea

ARTICLE INFO

Article history:
Received 24 April 2017
Received in revised form 14 December 2017
Accepted 20 January 2018
Available online xxxx

Keywords:
p-Laplace operator
Discrete p-Laplacian
Parabolic equation
Dichotomy

ABSTRACT

In this paper, we discuss and answer the following dichotomy problems: Let S be a network and $\Delta_{p,\omega}$ be a discrete p-Laplace operator with $1 < p < \infty$.

(i) If $u,\ v$ are functions satisfying

$$\begin{cases} -\Delta_{p,\omega}u(x) \le -\Delta_{p,\omega}v(x), & x \in S, \\ u(z) \le v(z), & z \in \partial S, \end{cases}$$

then either $u \equiv v$ on $\overline{S}$ or $u < v$ in S.

(ii) If $u,\ v$ are functions satisfying

$$\begin{cases} u_t(x,t) - \Delta_{p,\omega}u(x,t) \le v_t(x,t) - \Delta_{p,\omega}v(x,t), & (x,t) \in S \times (0,T), \\ u(x,0) \le v(x,0), & x \in S, \\ u(z,t) \le v(z,t), & (z,t) \in \partial S \times (0,T) \end{cases}$$

then either $u \equiv v$ on $\overline{S}$ or $u < v$ in $S \times (0,T)$.

We believe that this work is not only interesting in itself, but also gives a clue to solve the problems defined on the continuous domain.

1. Introduction

In the year 2013, there was held a workshop on the p-Laplace operators and their applications in which several open problems were proposed by P. Lindqvist and P. Takac in [1] as follows:

Let Ω be an open subset of $\mathbb{R}^N$ and $1 < p < \infty$.

Problem 1. If $u, v \in W_0^{1,p}(\Omega)$ are solutions to

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \text{ in } \Omega$$

satisfying $u \le v$ in Ω, then is it true that either $u \equiv v$ or $u < v$ in Ω?

Problem 2. If $u, v \in W_0^{1,p}(\Omega)$ are solutions to

$$\begin{cases} -\Delta_p u = f(x), & x \in \Omega,\ u|_{\partial\Omega} = 0, \\ -\Delta_p v = g(x), & x \in \Omega,\ v|_{\partial\Omega} = 0, \end{cases}$$

E-mail address: sychung@sogang.ac.kr.

https://doi.org/10.1016/j.camwa.2018.01.021

for $f, g \in L^\infty(\Omega)$ with $f \leq g$ $a.e.$ in Ω. If $f \not\equiv g$ in Ω, then is it true that

$$u < v \text{ in } \Omega \text{ and } \frac{\partial u}{\partial n} > \frac{\partial v}{\partial n} \text{ on } \partial\Omega?$$

Problem 3. Let $u, v : [0, T] \to W_0^{1,p}(\Omega)$ be (weak) solutions to

$$\begin{cases} u_t(x,t) - \Delta_p u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \\ v_t(x,t) - \Delta_p v(x,t) = g(x,t), & (x,t) \in \Omega \times (0,T), \\ v(x,0) = v_0(x), & x \in \Omega, \\ u(z,t) = v(z,t) = 0, & (z,t) \in \partial\Omega \times (0,T), \end{cases}$$

for $f, g \in L^\infty(\Omega \times (0,T))$ with $f \leq g$ $a.e.$ and $u_0, v_0 \in W_0^{1,p}(\Omega)$ with $u_0 \leq v_0$ $a.e.$

(i) If $u_0 \not\equiv v_0$ in Ω, then is it true that

$$u(x,t) < v(x,t) \text{ in } \Omega \times (0,T) \text{ and } \frac{\partial u}{\partial n}(x,t) > \frac{\partial v}{\partial n}(x,t) \text{ on } \partial\Omega \times (0,T)?$$

(ii) If $f \not\equiv g$ in $\Omega \times (0,T)$, then is (i) true?
(iii) If $f(\cdot,t) \not\equiv g(\cdot,t)$ in Ω for each $t \in (0,T)$, then is (i) true?

Besides these problems, some more questions were proposed there for more general nonlinearities in Problem 3.

As mentioned in [1], only special cases have been discussed and proved so far by several authors (see [2] and [3], for example) but most of those have been unsolved yet.

In the theory of the thermal propagation and combustion, it is natural to consider the following equation of the form

$$u_t(x,t) = \operatorname{div}\left(|\nabla u(x,t)|^{p-2}\nabla u(x,t)\right) + f(x,t)$$

as a nonlinear heat propagation model (see [4] and [5]). Here, u and the term $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ stand for the temperature and the diffusion of heat over a non-Newtonian medium, respectively. In addition, the quantity p is a characteristic of the medium; for example, media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics; if $p = 2$, they are Newtonian (see [6]).

The purpose of this paper is to set up discrete analogue of each problem above and to give an answer to it. To be precise, introducing the discrete p-Laplace operators and the discrete p-Laplace heat equations, we give answers to Problems 1, 2 and 3-(iii) positively and Problem 3-(ii) negatively, by some counter-examples. On the other hand, Problem 3-(i) is going to be answered negatively for the case $1 < p < 2$ and positively for the case $p \geq 2$.

Even though, we discussed here the above open problems only in the discrete settings, instead of the continuous settings, we believe that our results are not only interesting in itself, but also give a clue on how we expect the behavior of the solutions to the continuous cases, since the continuous case is basically approximated by the discrete case by way of numerical schemes.

This paper is organized as follows: After introducing graph theoretic notations in Section 2, we discuss and give an answer to Problem 1 in Section 3. Section 4 is devoted to discuss and answer Problems 2 and 3.

2. Preliminaries and discrete calculus

Here, we introduce preliminary concepts for a weighted graph and basic discrete calculus on a weighted graph.

By a graph $G := G(V, E)$, we mean a finite set V (whose elements are called vertices) along with a set E consisting of line segments whose end-points are elements of V (the elements of E are called edges).

A weighted graph (or a network) $G := G(V, E, \omega)$ is a graph $G := G(V, E)$ associated with a weight function $\omega : V \times V \to [0, \infty)$ satisfying that

(i) $\omega(x,x) = 0, x \in V$,
(ii) $\omega(x,y) = \omega(y,x)$, for all $x, y \in V$
(iii) $\omega(x,y) > 0$ if and only if $x \sim y$

where $x \sim y$ means that two vertices x and y are connected (adjacent) by an edge in E.

In this paper a graph G is assumed to be simple, i.e. it has neither multiple edges nor loops.

By a graph S with boundary ∂S, we mean a subgraph $S := S(V', E')$ of a weighted graph $G := G(V, E, \omega)$ such that $V' \subset V$ and $E' \subset E$ and the end-point of each edge in E' belongs to V' and ∂S is the set of vertices such that each $z \in \partial S$ is adjacent to some $x \in S$. Furthermore, in this case, we write as $\overline{S} := S \cup \partial S$.

Throughout this paper, we always assume that S is connected, i.e. for any pair of vertices x and y in S, there exist vertices $x_1, x_2, \ldots, x_n \in S$ such that

$$x \sim x_1 \sim x_2 \sim \ldots \sim x_n \sim y.$$

Please cite this article in press as: S.-Y. Chung, Dichotomy of solutions to discrete p-Laplace equations and p-Laplace parabolic equations, Computers and Mathematics with Applications (2018), https://doi.org/10.1016/j.camwa.2018.01.021.