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Radial basis function partition of unity method for modelling water flow in porous media

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Water flow in variably-saturated porous media is modelled by using the highly nonlinear parabolic Richards' equation. The nonlinearity is due to the hydraulic conductivity and moisture content variables. The latter were estimated by using experimental models, including Gardner, Burdine, Mualem and van Genuchten models. The aim of this work is to develop a new technique based on the radial basis function partition of unity method (RBF-PUM) and Gardner model in order to solve Richards' equation in one and two dimensions. We have used Gardner model to handle the nonlinearity issue and the RBFPUM is used to approximate the solution of the linearized Richards' equation. Our proposed algorithm is based on testing many configurations of the partitions number and selecting the optimal shape parameter for each case. Then we pick up the optimal configuration (partitions number-shape parameter) that yields the best solution in terms of error and conditioning number. By following this procedure, an optimal solution is ensured for our given problem. As numerical tests, we consider the vertical infiltration of water in soils in order to validate our proposed method.

1. Introduction

Studying water flow in variably-saturated porous media is an important issue in hydrology and many other engineering sciences. In fact, many phenomena happen in the vadose zone located between ground surface and water table. Examples of these phenomena are infiltration, groundwater discharge, evapotranspiration... Thus, studying water movement in the vadose zone becomes important in order to understand our underwater resources and for many other agricultural reasons. Richards' equation [1] is used for modelling the transport of water in a variably-saturated homogeneous and isotropic porous media. It has been derived from combining the generalized Darcy's law and the principle of mass conservation [1]. Its popular formulation based on pressure head variable is considered here. It is a highly nonlinear parabolic equation with diffusion and advection terms. That is why analytical solutions are impossible to obtain except for some problems with special initial and boundary conditions such as the case in [2–5]. The nonlinearity has been handled by using an iterative algorithm such as Picard, Newton or modified Picard iterations [6,7]. However, in that case, bad numerical solutions have been obtained. Furthermore, the method used has been cost numerically due to the nonlinearity of this equation and the complex models of hydraulic conductivity [8–10]. Many authors were motivated to refer to a simpler form of hydraulic conductivity called the exponential formulation of Gardner's model [11]. The idea was to make a change of variable in order to linearize Richards' equation. For this purpose, two similar approaches were adopted. The first one was based on considering the exponential model of hydraulic conductivity as a novel variable [2], while the second one used the Kirchhoff transformation in order

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to eliminate the nonlinearity of the diffusion term [12,13]. The first approach is considered here. It does not only allow to compute the hydraulic conductivity but also the pressure head term which is the main variable of Richards' equation.

The aim of this work is to develop a new efficient algorithm based on the radial basis function partition of unity method (RBFPUM) in order to solve Richards' equation in one and two dimensions. The RBFPUM is a local meshless method introduced in [14] and based on summing the local solutions depending on their weights. These local solutions are written in the basis of radial functions (RBFs) and the weight functions are considered as locally supported RBFs. RBFPUM can be very competitive to the already existing local methods. Preconditioning with RBFPUM has been later discussed in [15]. Besides, some option pricing problems were solved using this method in [16,17]. In [16], a comparison between global RBF and RBFPUM has been achieved for the option pricing problems. It has been demonstrated that the RBFPUM is the most efficient method for these problems. The algorithm proposed in this paper is based on making compromise between a good conditioning number and the best error by testing many configurations and selecting the optimal number of subdomains and its associated shape parameter among all these configurations. The highly nonlinear parabolic Richards' equations in one and two dimensions are linearized first and the linearized equations are solved by using the global RBF and the algorithm introduced here. The tests considered in this paper are the Green–Ampt problems with well known analytical solutions [18]. The results will show the efficiency of the proposed algorithm.

This paper is structured as follows. In Section 2, we introduce Richards' equation in one and two dimensions. Furthermore, we show how we linearize this equation in order to be easily solved. Then, in Section 3, we give an overview of global RBF method as well as a discretization of linearized Richards' equation using this method. Section 4 contains the background of RBFPUM with discretization of the linearized Richards' equation. In Section 5, numerical results are shown with some discussions about our proposed algorithm. Finally, we make some conclusions in Section 6.

2. Richards' equation

2.1. Model

Richards' equation [1] is used for modelling the groundwater flow in a homogeneous and isotropic soil. It was derived by combining the generalized Darcy's law $q = -K(h)\nabla H$, which describes the proportionality between infiltration flux q and hydraulic conductivity $K(h)$, and conservation of the mass

$$\frac{\partial \theta}{\partial t} + \nabla . q = S(\mathbf{x}),$$

where $\nabla .$ is the divergence and $S(\mathbf{x})$ is a source/sink term.

This combination leads to the general Richards' equation:

$$\frac{\partial \theta}{\partial t} = \nabla . \left(K(h)\nabla H\right) + S(\mathbf{x}), \quad \mathbf{x} \in \Omega,\ 0 < t \leq T, \tag{1}$$

where h is the pressure head, $H = h + z$ is the total head, z is the position in an ascendent axis, θ is the volumetric water content of the porous media, $K(h)$ is the hydraulic conductivity, Ω is an open set of $\mathbb{R}^d$ and t is time variable less than a maximum value T.

The one dimensional Richards' equation can be derived

$$C(h)\frac{\partial h}{\partial t} = \frac{\partial}{\partial z}\left(K(h)\frac{\partial H}{\partial z}\right) + S(z), \quad z \in \mathbb{R},\ 0 < t \leq T, \tag{2}$$

and the similar two dimensional Richards' equation is

$$C(h)\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K(h)\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial z}\left(K(h)\frac{\partial H}{\partial z}\right) + S(\mathbf{x}), \tag{3}$$

where $\mathbf{x} = (x, z) \in \mathbb{R}^2$ is the space variable, z is the vertical axis positive upward and $C(h) = \frac{d\theta}{dh}$ is the capillary capacity.

The nonlinear terms $C(h)$ and $K(h)$ do not allow to obtain an efficient solution. Therefore, it is necessary to make some linearizing techniques that can handle the nonlinearity issue.

In order to illustrate the linearizing technique, we focus on the two dimensional Richards' equation (3). In the same way we can treat the one dimensional case by just omitting the terms with the first component x. In the next paragraph, we will see linearizing procedure.

2.2. Linearizing technique

The idea here is to transform the Richards' equation and its associated boundary and initial conditions to a PDE where the main variable is the hydraulic conductivity model developed by Gardner [4,2,13,3,18]. More precisely, we make the following change of variable:

$$K(h) = k_s e^{\alpha h}. \tag{4}$$