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Fractional Wishart processes and ε-fractional Wishart processes with applications[☆]

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ABSTRACT

In this paper, we introduce two new matrix stochastic processes: fractional Wishart processes and ε-fractional Wishart processes with integer indices which are based on the fractional Brownian motions and then extend ε-fractional Wishart processes to the case with non-integer indices. Both processes include classic Wishart processes (if the Hurst index H equals $\frac{1}{2}$) and present serial correlation of stochastic processes. Applying ε-fractional Wishart processes to financial volatility theory, the financial models account for the stochastic volatilities of the assets and for the stochastic correlations not only between the underlying assets' returns but also between their volatilities and for stochastic serial correlation of the relevant assets.

1. Introduction

Since Black and Scholes' significant work [1], more and more stochastic processes are widely used to capture diverse phenomena in financial markets, such as Brownian motions, fractional Brownian motions, Lévy processes, Wishart processes and so on.

Heston's model [2] adopts Brownian motions to describe the stochastic volatility of the stock, as empirical evidences [3,4] have shown that the classic Black–Scholes assumption of lognormal stock diffusion with constant volatility is not consistent with the market price (such phenomenon is often referred to as the volatility skew or smile).

As the backbone of multivariate statistical analysis, random matrices have found their applications in many fields, such as physics, economics, psychology and so on. Bru [5] develops Wishart processes in mathematics which are dynamic random matrices and turn out to be a better way to capture stochastic volatility and correlation structure of the relevant random vectors. In recent years, there has been tremendous growth of multi-asset financial contracts (outperformance options, for example), which exhibit sensitivity to both the volatilities and the correlations of the underlying assets. The authors of [6] and [7] show that the correlations between financial assets evolve stochastically and are far from remaining static through time. Furthermore, in [8] and [9], there are evidences which present that the higher the market volatility is, the higher the correlations between financial assets tend to be. In order to include those phenomena in financial markets, Wishart Affine Stochastic Correlation models [10–12] introduce Wishart processes to account for the stochastic volatilities of the assets and for the stochastic correlations not only between the underlying assets' returns but also between their volatilities.

There also exists early evidence [13] which shows the processes of observable market values seem to exhibit serial correlation (this means the increments of the process depend on the information of the past). So fractional Brownian motions

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(including mixed fractional Brownian motions) are proposed for mapping this kind of behavior. Fractional Brownian motions not only capture serial correlation of stochastic processes, but also keep a good analytical treatability for still being Gaussian, which leads that they become an interesting and important candidate for financial models. Furthermore, generalized results of fractional Brownian motions often include the corresponding well-known results of classic Brownian motions, as fractional Brownian motions are just classic Brownian motions when the Hurst index equals $\frac{1}{2}$. Mandelbrot and van Ness [14] suggested fractional Brownian motions as alternative models for assets' dynamics which allow for dependence between returns over time. Shokrollahi and Kılıçman [15] provided a new framework for pricing currency options in accordance with the fractional Brownian motion model to capture long-memory property of the spot exchange rate. Sun [16] He and Chen [17] used financial models with mixed fractional Brownian motions to capture the long-range dependence of the stock returns effectively for currency options and credit default swaps, respectively. Nowadays there is an ongoing dispute on the usage of (mixed) fractional Brownian motions in financial theories [18–21].

In our paper, we shall introduce two new matrix stochastic processes: the fractional Wishart process and ε-fractional Wishart process. First of all, the fractional Wishart process (which is based on the fractional Brownian motion) is the generalization of the Wishart process (which is based on the classic Brownian motion) such that the former degenerates to the latter when the Hurst index equals $\frac{1}{2}$. The fractional Wishart process can present serial correlation of stochastic processes while the Wishart process is a Markov process (see Definition 2 in [22]) whose increments are independent of the past such that the Wishart process can be thought of as 'memoryless'. The ε-fractional Wishart process is the approximation of the fractional Wishart process as the fractional Wishart process does not keep a good analytical treatability. The difference between the Wishart process and ε-fractional Wishart process is that the former process is governed by a related stochastic differential equation (SDE) while the latter process is governed by a related stochastic partial differential equation (SPDE). Of course, the ε-fractional Wishart process becomes the Wishart process when the Hurst index equals $\frac{1}{2}$. In financial theory, if we apply the fractional Wishart process or ε-fractional Wishart process to the volatility of assets, then the model shall account for the stochastic volatilities of the assets and for the stochastic correlations not only between the underlying assets' returns but also between their volatilities and for stochastic serial correlation of the relevant assets.

The rest of the paper is organized as follows. Section 2 sketches the main assumptions and results needed in this paper. Then we define the fractional Wishart process with an integer index in Section 3. In Section 4, we define ε-fractional Wishart process with an integer index and then extend ε-fractional Wishart process to the case with a non-integer index by its related SPDE. In Section 5, a generalization of the ε-fractional Wishart process which includes two more parameters is discussed. In Section 6, we apply the ε-fractional Wishart process to financial volatility model. Finally, some conclusions and future work are included in Section 7.

2. Preliminaries

In this section, we shall sketch some basic concepts related to matrix variate distributions and fractional Brownian motions and so on (one can refer to [23–25] for details).

Let $(\mathbf{\Omega}, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P}_0)$ be a filtered probability space satisfying the usual conditions. The stochastic processes are considered in such probability space if we do not give the probability space.

For any positive integers n and p, let $\mathcal{M}_{n,p}(\mathbb{R})$ (resp. $\mathcal{S}_p(\mathbb{R})$, $\mathcal{S}_p^+(\mathbb{R})$ and $\overline{\mathcal{S}_p^+}(\mathbb{R})$) denote the sets of all real-valued $n \times p$ matrices (resp. $p\times p$ symmetric matrices, $p\times p$ symmetric positive definite matrices and $p\times p$ symmetric positive semidefinite matrices). For any real Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, let $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ denote the space of bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$. In $\mathcal{M}_{n,p}(\mathbb{R})$ the norm of any matrix $A = (A_{ij})_{1\leq i\leq n, 1\leq j\leq p}$, denoted by $|A|_{n\times p}$ or just $|A|$ is defined as the Frobenius norm (see, for example, [26]):

$$|A| = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{p}|A_{ij}|^2}.$$

For normed spaces $\mathbb{X}_1$ and $\mathbb{X}_2$, we will define the norm on $\mathbb{X}_1 \times \mathbb{X}_2$ as $|\cdot|_{\mathbb{X}_1} + |\cdot|_{\mathbb{X}_2}$ (or $|\cdot| + |\cdot|$). For any map $f(x_1, \ldots, x_n)$, let $\partial_{x_i}^k f$ denote the kth partial derivative of f and $\partial_{x_1}^k f$ be denoted by $\partial^k f$ when $n = 1$. Moreover, we always use X' to denote the transpose of a matrix X.

For a definition of a random matrix, as well as its probability density function (p.d.f.), the moment generating function (m.g.f) and so on, one can refer to [23]. For any random matrix $X \in \mathcal{M}_{n,p}(\mathbb{R})$, we mean that the random matrix X takes its values in the set $\mathcal{M}_{n,p}(\mathbb{R})$.

Let $A \in \mathcal{M}_{m,n}(\mathbb{R})$ and $B \in \mathcal{M}_{p,q}(\mathbb{R})$. Then the Kronecker product (also called the direct product, see, for example, [23]) of A and B, denoted by $A \otimes B$, is defined by

$$A \otimes B = (A_{ij}B) = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{21}B & \cdots & A_{11}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m1}B & \cdots & A_{mn}B \end{pmatrix}.$$

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