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Variants of the deteriorated PSS preconditioner for saddle point problems☆

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ABSTRACT

Two new preconditioners, which can be viewed as variants of the deteriorated positive definite and skew-Hermitian splitting preconditioner, are proposed for solving saddle point problems. The corresponding iteration methods are proved to be convergent unconditionally for cases with positive definite leading blocks. The choice strategies of optimal parameters for the two iteration methods are discussed based on two recent optimization results for extrapolated Cayley transform, which result in faster convergence rate and more clustered spectrum. Compared with some preconditioners of similar structures, the new preconditioners have better convergence properties and spectrum distributions. In addition, more practical preconditioning variants of the new preconditioners are considered. Numerical experiments are presented to illustrate the advantages of the new preconditioners over some similar preconditioners to accelerate GMRES.

1. Introduction

Consider the following saddle point problem:

$$\mathscr{A}_+ X \equiv \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv c_+, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a positive definite or positive semidefinite matrix, $B \in \mathbb{R}^{m\times n}$ is a rectangular matrix with $m < n, f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ are given vectors. We assume that $\mathscr{A}_+$ is nonsingular, which implies that B is of full row rank. Especially, when A is symmetric positive semidefinite, then a necessary and sufficient condition for $\mathscr{A}_+$ to be nonsingular is $\mathscr{N}(A) \cap \mathscr{N}(B) = \{0\}$. Here, $\mathscr{N}(\cdot)$ denotes the null space of a matrix. Negativing the second block row of (1.1) equivalently yields

$$\mathscr{A} X \equiv \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv c. \tag{1.2}$$

This nonsymmetric formulation is especially natural when A is nonsymmetric, but positive real. In fact, $\mathscr{A}$ is positive stable, i.e., the eigenvalues of $\mathscr{A}$ have nonnegative real parts; see [1]. This can be advantageous when using Krylov subspace methods.

The augmented linear system (1.1) or (1.2) can be derived from many scientific computing and engineering applications, such as computational fluid dynamics, electromagnetics, structural mechanics, interior point methods, constrained and

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weighted least square problems, interpolation of scattered data, model order reduction and optimal control; see [2,3] and the references therein.

Numerous efficient iteration methods have been proposed for solving the linear system (1.1) or (1.2), such as splitting based methods [4–7], inexact and parameterized Uzawa methods [8–10], and so on. In addition, iteration methods such as Krylov subspace methods [11–13] are particularly attractive. However, the rate of convergence of these methods without preconditioning may be very slow. Many preconditioners for saddle point matrices have been proposed, for example, block diagonal and block triangular preconditioners [14–16], constraint preconditioners [17–19], augmented Lagrangian preconditioners [20,21], Hermitian and skew-Hermitian splitting (HSS) preconditioners [1,22,23], deteriorated positive definite and skew-Hermitian splitting (DPSS) preconditioner and its variants [24–29], and so on.

In [1,30], Bai et al. showed that the coefficient matrix $\mathscr{A}$ in (1.2) has the HSS:

$$\mathscr{A}=\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix}=\begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} S & B^T \\ -B & 0 \end{pmatrix}\equiv\mathscr{H}+\mathscr{S}, \tag{1.3}$$

where $H=\frac{1}{2}(A+A^T)$ and $S=\frac{1}{2}(A-A^T)$ are the Hermitian and skew-Hermitian parts of A, respectively. Then we have the following HSS iteration scheme

$$\begin{cases}(\alpha I+\mathscr{H})X^{(k+\frac{1}{2})}=(\alpha I-\mathscr{S})X^{(k)}+c,\\ (\alpha I+\mathscr{S})X^{(k+1)}=(\alpha I-\mathscr{H})X^{(k+\frac{1}{2})}+c,\end{cases} \tag{1.4}$$

where $\alpha > 0$ and I denotes the identity matrix of proper size. The iteration scheme (1.4) can result in the splitting $\mathscr{A}=\mathscr{P}_{\text{HSS}}-(\mathscr{P}_{\text{HSS}}-\mathscr{A})$ with the HSS preconditioner

$$\mathscr{P}_{\text{HSS}}=\frac{1}{2\alpha}(\alpha I+\mathscr{H})(\alpha I+\mathscr{S})=\frac{1}{2\alpha}\begin{pmatrix} \alpha I+H & 0 \\ 0 & \alpha I \end{pmatrix}\begin{pmatrix} \alpha I+S & B^T \\ -B & \alpha I \end{pmatrix}. \tag{1.5}$$

The HSS preconditioner is an effective preconditioner for solving the saddle point problem (1.2). In [1], convergence of the HSS iteration scheme (1.4) and spectral properties of the preconditioned matrix $\mathscr{P}_{\text{HSS}}^{-1}\mathscr{A}$ are established.

Moreover, the coefficient matrix $\mathscr{A}$ in (1.2) has the deteriorated positive definite and skew-Hermitian splitting (DPSS) [24]:

$$\mathscr{A}=\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix}=\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & B^T \\ -B & 0 \end{pmatrix}\equiv\mathscr{P}+\tilde{\mathscr{S}}. \tag{1.6}$$

Then we have the following DPSS iteration scheme

$$\begin{cases}(\alpha I+\mathscr{P})X^{(k+\frac{1}{2})}=(\alpha I-\tilde{\mathscr{S}})X^{(k)}+c,\\ (\alpha I+\tilde{\mathscr{S}})X^{(k+1)}=(\alpha I-\mathscr{P})X^{(k+\frac{1}{2})}+c,\end{cases} \tag{1.7}$$

where $\alpha > 0$. The iteration scheme (1.7) can result in the splitting $\mathscr{A}=\mathscr{P}_{\text{DPSS}}-(\mathscr{P}_{\text{DPSS}}-\mathscr{A})$ with the DPSS preconditioner

$$\mathscr{P}_{\text{DPSS}}=\frac{1}{2\alpha}(\alpha I+\mathscr{P})(\alpha I+\tilde{\mathscr{S}})=\frac{1}{2\alpha}\begin{pmatrix} \alpha I+A & 0 \\ 0 & \alpha I \end{pmatrix}\begin{pmatrix} \alpha I & B^T \\ -B & \alpha I \end{pmatrix}. \tag{1.8}$$

This preconditioner is a valid variant of the HSS preconditioner (1.5) and it will reduce to (1.5) when A in (1.2) is symmetric. In [31], convergence theory of the DPSS iteration scheme (1.7) is obtained. It is proved to be convergent for all $\alpha > 0$. Moreover, eigenvalue distributions of the preconditioned matrix $\mathscr{P}_{\text{DPSS}}^{-1}\mathscr{A}$ is discussed in [24].

The HSS and DPSS preconditioners (1.6) and (1.8) are both parameter dependent, so the selection of optimal parameters is crucial for their implementations. However, it is a tough thing for the selection of optimal iteration parameters. The optimal or quasi-optimal iteration parameters of the HSS iteration scheme (1.4) for solving the saddle point problem (1.2) with symmetric positive definite (1,1) block are discussed in [4,23,32]. In [4], the optimal iteration parameter is obtained for (1.2) with (1,1) block being equivalent to the identity matrix. In [32], the quasi-optimal iteration parameter is obtained for the more practical but more difficult case that the (1,1) block is not algebraically equivalent to the identity matrix. In [23], the optimization of the HSS iteration is realized by Fourier transforms for simple model problem of the Poisson equation. However, for general case of (1.2) with (1,1) block being nonsymmetric, no optimization results exist for the HSS and DPSS preconditioners. Besides, it is known that the HSS iteration scheme (1.4) and the DPSS iteration scheme (1.7) are both convergent unconditionally for saddle point problem (1.2) with positive definite (1,1) block, but the unconditional convergence results of the two iteration schemes no longer hold for the case with positive semidefinite (1,1) block.

In this paper, we restrict our attentions to the two-by-two block structure of the saddle point problem (1.2), but not to the information about underlying system of the partial differential equations. Two new matrix splitting preconditioners are proposed, which can be view as improved variants of the DPSS and HSS preconditioners. When the leading block of (1.2) is positive definite, the corresponding iteration methods are proved to be convergent unconditionally to its exact solution. Spectral properties of the corresponding preconditioned matrices are investigated, results about eigenvalue and eigenvector distributions and upper bounds of the degrees of the minimal polynomials are obtained. The optimal parameters resulting in faster convergence rate and more clustered spectrum for the two preconditioners are also deduced, which are based