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An improved tau method for the multi-dimensional fractional Rayleigh–Stokes problem for a heated generalized second grade fluid

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HIGHLIGHTS

- Two efficient algorithms for the multi-dimensional fractional Rayleigh–Stokes problem are proposed.
- The proposed algorithms are based on the Legendre spectral tau method.
- Discussions on the L^2-convergence of the method are presented.
- The theoretical results are verified by numerical experiments.

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ABSTRACT

We develop efficient algorithms based on the Legendre-tau approximation for one- and two-dimensional fractional Rayleigh–Stokes problems for a generalized second-grade fluid. The time fractional derivative is described in the Riemann–Liouville sense. Discussions on the L^2-convergence of the proposed method are presented. Numerical results for one- and two-dimensional examples with smooth and nonsmooth solutions are provided to verify the validity of the theoretical analysis, and to illustrate the efficiency of the proposed algorithms.

1. Introduction

Recently, fractional differential equations have stimulated renewed interest both in mathematics and in applications due to their enormous potential to model complex multi-scale problems and anomalous transport phenomena. Linear viscoelasticity is certainly the field of the most extensive applications of fractional calculus since its appearance, in view of its ability to model hereditary phenomena with cumulative memory (see, e.g., [1–3]).

The aim of this study is to revisit a Legendre-tau method to handle the multi-dimensional fractional Rayleigh–Stokes problem for a heated generalized second-grade fluid. Let $\Omega \subset \mathbb{R}^d$ ($d = 1, 2$) be a bounded domain with its boundary $\partial\Omega$, $\mathcal{I} = (0, T]$ and $T > 0$. Then the mathematical model is described by the equation:

$$\begin{aligned} &\frac{\partial \mathscr{U}}{\partial t} = \left(\xi + \eta\, {}_0^{RL}D_t^{1-\mu}\right) \Delta \mathscr{U} + q, \quad \text{in } \Omega \times \mathcal{I}, \\ &\mathscr{U} = g, \quad \text{on } \partial\Omega \times \mathcal{I}, \\ &\mathscr{U}(\cdot, 0) = f, \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

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where ξ, η are two nonnegative constants, f and g are the initial and boundary data, and ${}^{RL}_{0}D_t^{1-\mu}$ is the Riemann–Liouville fractional derivative of order $1-\mu$, $\mu \in (0,1)$ defined by [1]

$$ {}^{RL}_{0}D_t^{1-\mu}\mathscr{U}(\mathbf{x},t) = \frac{\partial}{\partial t}\,{}_0I_t^{\mu}\mathscr{U}(\mathbf{x},t), \tag{1.2}$$

where ${}_0I_t^{\mu}$ denotes the Riemann–Liouville fractional integral of order μ defined by [2]

$$ {}_0I_t^{\mu}\mathscr{U}(\mathbf{x},t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\xi)^{\mu-1}\mathscr{U}(\mathbf{x},\xi)\,d\xi. \tag{1.3}$$

The fractional Rayleigh-Stokes problem (1.1) plays a significant role in describing the dynamic behavior of some non-Newtonian fluids. The presence of the time-fractional derivative in the equation to capture the viscoelastic behavior of the flow [4,5]. In order to gain insights into the behavior of the solution of this model, there has been substantial interest in deriving a closed form solution [6]. Shen et al. [7] derived exact solutions of this model using the Fourier transform and the fractional Laplace transform. Xue and Nie [8] obtained also closed form solutions of this model in a porous half-space using both Fourier and fractional Laplace transforms. Zhao and Yang [9] obtained exact solutions using the eigenfunction expansion.

The analytical solutions obtained in these studies involve special functions and infinite series, and thus are inconvenient for numerical evaluation. Hence, efficient and accurate numerical approaches are demanded to deal with this model. Chen et al. [10] developed explicit and implicit finite difference methods and discussed their stability and convergence using the Fourier method. Wu [11] developed an implicit numerical scheme by transforming the above mentioned problem into an integral equation. Lin and Jiang [12] introduced a numerical scheme based on the reproducing kernel space. Mohebbi et al. [13] compared a fourth-order compact scheme with radial basis functions meshless approach. Recently, Bazhlekova et al. [14] developed two fully discrete schemes based on the backward difference method and backward Euler method and a semidiscrete scheme based on the Galerkin finite element method. Bhrawy et al. [15] developed two shifted Jacobi–Gauss collocation schemes. More recently, Dehghan and Abbaszadeh [16] developed a finite element method for two-dimensional fractional Rayleigh–Stokes model on complex geometries. Shivanian and Jafarabadi [17] developed a spectral meshless radial point interpolation technique. Abu Arqub [18] introduced a novel solver based on the reproducing kernel Hilbert space method for treating this model with Neumann boundary conditions. Chen et al. [19] proposed a numerical algorithm based on the second-order compact approximation of first-order derivative with second-order temporal accuracy and fourth-order spatial accuracy.

Numerical approaches for fractional differential equations have been investigated for over two decades [1,20–23]. The key challenge in developing these methods is the large computational cost due to the global (nonlocal) nature of fractional differential operators. This challenge would suggest that global schemes such as spectral methods are more appropriate tools for discretizing fractional differential equations, especially in the case of smooth solutions [24]. There is limited but very promising research on developing spectral and pseudospectral methods for solving fractional differential equations [25–30]. Among these types of spectral methods are tau [31–35], Galerkin [36,37], and collocation methods [38–41]. To the best of our knowledge, the Legendre-tau method has not been applied to the fractional Rayleigh–Stokes problem (1.1).

The paper is structured as follows. After a section on describing the governing equation and the operational matrices for differentiation and fractional integration of the shifted Legendre polynomial vector, the Legendre-tau method in one-dimensional case is developed in Section 3. The convergence of the method is proved in Section 4. In Section 5, the two-dimensional fractional Rayleigh–Stokes problem is discussed. In Section 6, two benchmark examples are considered for comparing the present algorithms with the results reported in the literature. Finally, some conclusions are given in Section 7.

2. Fundamentals

2.1. Constitutive equations

The incompressible second grade fluid model has been found to be useful in describing the behavior of certain fluids which cannot be adequately described by the classical Newtonian model. The stress T in such a fluid is given by [4,42]

$$ \mathsf{T} = \mu\mathsf{A}_1 - \rho\mathsf{I} + \alpha_1\mathsf{A}_2 + \alpha_2\mathsf{A}_1^2, \tag{2.1}$$

where

- I is the identity tensor,
- ρ is the hydrostatic pressure,
- μ is μ the viscosity of the fluid,
- α_1, α_2 are normal stress moduli,
- A_1, A_2 are the kinematic tensors defined through