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Inhomogeneous Dirichlet boundary condition in the *a posteriori* error control of the obstacle problem

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ABSTRACT

We study *a posteriori* error control of finite element approximation of the elliptic obstacle problem with nonhomogeneous Dirichlet boundary condition. The results in the article are two fold. Firstly, we address the influence of the inhomogeneous Dirichlet boundary condition in residual based *a posteriori* error control of the elliptic obstacle problem. Secondly by rewriting the obstacle problem in an equivalent form, we derive *a posteriori* error bounds which are in simpler form and efficient. To accomplish this, we construct and use a post-processed solution $\tilde{u}_h$ of the discrete solution u_h which satisfies the exact boundary conditions sharply although the discrete solution u_h may not satisfy. We propose two post processing methods and analyze them, namely the harmonic extension and a linear extension. The theoretical results are illustrated by the numerical results.

1. Introduction

The elliptic obstacle problem is a popular prototype model for the study of elliptic variational inequalities. The applications of variational inequalities are enormous in the modern scientific computing world, e.g. in contact mechanics, option pricing and fluid flow problems. The numerical analysis of these classes of problems is an interesting subject as they offer challenges both in theory and computation. We refer to the books [1–4] for the theory of variational inequalities and their corresponding numerical analysis. Apart from these, we refer to the articles [5,6] and the recent articles [7–11] for the convergence analysis of finite element methods for the obstacle problem. One of the interesting properties that the obstacle problem exhibits is the free boundary along which the regularity of the solution is affected. It is worth remarking here that the location of free boundary is not known *a priori*. Adaptive finite element methods based on reliable and efficient *a posteriori* error estimates are of particular interest in this contest as they can capture the free boundaries by local mesh refinement around them. In designing any adaptive scheme, the first step is to derive some computable error estimators which are both reliable and efficient, see [12] for the analysis of *a posteriori* error control. There are many works in deriving residual based *a posteriori* error estimates for the obstacle problem, see [13–24] and see [25–28]. In recent years, much of research is focused on proving the convergence of adaptive methods based on *a posteriori* error estimates. In this direction, we refer to [29–34] for the work related to the obstacle problem. Further, we refer [35–38] and [39–42] for the work related to the numerical approximation of the Signorini contact problem.

In many occasions, it is assumed for the convenience in *a posteriori* error analysis of obstacle problems that the Dirichlet data is either zero or trace of a finite element function. However it is not clear if the error estimator with homogeneous Dirichlet boundary condition is reliable and efficient in the energy norm up to some Dirichlet data oscillations. The answer

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to this question so far seems to be not clear as it can be seen in [32, Section 4.2] that the error estimator is proved only weakly reliable with nonhomogeneous Dirichlet boundary data oscillations. Exceptions to this question hold for the local error analysis (estimates in maximum norm) in [21,43] where reliable and efficient error estimates were derived for general Dirichlet boundary data. One of the difficulties in the energy norm error estimate arises due to the fact that the error $u - u_h$ does not belong to $H_0^1(\Omega)$. It will be difficult to argue with the residual directly using the error $u - u_h$. We may think of introducing an auxiliary problem correcting this as in the case of linear elliptic problems, but with this the estimator may consists of the unknown solution of that auxiliary problem. In this article, we introduce two methods namely harmonic extension (see [44,45]) and linear extension for constructing a post-processed solution in which the later can be computed explicitly. We also derive some estimates to quantify the error between the discrete and its post-processed solution. It is worth mentioning here that both the extensions are positive preserving and the linear extension is optimally convergent in L^2-norm as well. Subsequently, we address the question of proving the reliability and efficiency of the error estimator up to some Dirichlet data oscillations as in the case of linear elliptic problems. The results also can be viewed in another aspect that since the obstacle problem with a general obstacle χ can be transformed into a problem with zero obstacle with nonhomogeneous Dirichlet boundary condition, for example see [32], the error estimator for general obstacle problem can be simplified to the error estimator for the zero obstacle. Generally, the error estimator for zero obstacle problem is simpler, for example free of min/max functions, dealing with nonconformity in the approximation of obstacle constraint.

The rest of the article is organized as follows. In the remaining part of this section, we introduce the model problem and some preliminaries. In Section 2, we introduce some notation, define the discrete problem and derive some properties of the discrete solution. In Section 3, we construct a post-processed discrete solution. We propose two methods for this purpose. One of them is by harmonic extension and the other by linear extension. Therein, we derive some error estimates for the discrete solution and its post-processed solution. In Section 4, we present the *a posteriori* error analysis. In Section 5, we derive error bounds that are simpler by rewriting general obstacle problem into a problem with zero obstacle. We present some numerical experiments in Section 6 to illustrate the theoretical findings and finally conclude the article in Section 7.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary denoted by $\partial\Omega$ (without slit). However the results on harmonic extension and the results in Section 5 are applicable to three dimensional problems. We consider the obstacle $\chi \in C(\bar{\Omega}) \cap H^1(\Omega)$ satisfying $\chi|_{\partial\Omega} \le g$ a.e. on $\partial\Omega$, hereafter the function g is assumed to be given and denotes the Dirichlet boundary data. Further, we assume that g is the trace of a $H^1(\Omega)$ function. Define the closed and convex set by

$$\mathcal{K} := \{v \in H^1(\Omega) : v \ge \chi \text{ a.e. in } \Omega, \quad \gamma_0(v) = g \text{ on } \partial\Omega\},$$

where $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace map, whose range is denoted by $\tilde{H}^{1/2}(\partial\Omega)$. Since $g \in \tilde{H}^{1/2}(\partial\Omega)$, there is some $\tilde{g} \in H^1(\Omega)$ with $\gamma_0(\tilde{g}) = g$. Then it can be seen that the set $\mathcal{K}$ is nonempty as $\chi^+ := \max\{\chi, \tilde{g}\} \in \mathcal{K}$. The model problem for the discussion consists of finding $u \in \mathcal{K}$ such that

$$a(u, v - u) \ge (f, v - u) \quad \text{for all } v \in \mathcal{K}, \tag{1.1}$$

where $a(u, v) := (\nabla u, \nabla v)$ and $f \in L^2(\Omega)$ is a given function. Hereafter, $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ inner-product while $\|\cdot\|$ denotes the $L^2(\Omega)$ norm. For any open set D, we denote by $\|\cdot\|_{H^m(D)}$ (resp. $|\cdot|_{H^m(D)}$) the standard norm (resp. semi-norm) on the Sobolev space $H^m(D)$, see [46]. The existence of a unique solution to (1.1) follows from the result of Stampacchia [1–3].

For the rest of the discussions, we assume that the Dirichlet data $g \in \tilde{H}^{1/2}(\partial\Omega) \cap C(\partial\Omega)$.
Define the Lagrange multiplier $\sigma \in H^{-1}(\Omega)$ by

$$\langle \sigma, v \rangle = (f, v) - a(u, v) \quad \text{for all } v \in H_0^1(\Omega), \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. The norm on $H^{-1}(\Omega)$ is denoted by $\|\cdot\|_{-1}$. It follows from (1.2) and (1.1) that

$$\langle \sigma, v - u \rangle \le 0 \quad \text{for all } v \in \mathcal{K}. \tag{1.3}$$

Note that in Eq. (1.3), the test function $v - u \in H_0^1(\Omega)$ and hence the duality bracket $\langle \sigma, v - u \rangle$ is meaningful.

2. Notation and preliminaries

Below, we list the notation that will be used throughout the article:

$\mathcal{T}_h$:= a regular simplicial triangulations of Ω

T := a triangle of $\mathcal{T}_h$, $\quad |T|$:= area of T

h_T := diameter of T, $\quad h := \max\{h_T : T \in \mathcal{T}_h\}$

$\mathcal{V}_h^i$:= set of all vertices of $\mathcal{T}_h$ that are in Ω

$\mathcal{V}_h^b$:= set of all vertices of $\mathcal{T}_h$ that are on $\partial\Omega$

$\mathcal{V}_T$:= set of three vertices of T

$\mathcal{E}_h^i$:= set of all interior edges of $\mathcal{T}_h$