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A splitting-based finite element method for the Biot poroelasticity system

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ABSTRACT

In this work, we propose a finite element method for solving the linear poroelasticity equations. Both displacement and pressure are approximated by continuous piecewise polynomials. The proposed method is sequential, leading to decoupled smaller linear systems compared to the systems resulting from a fully implicit finite element approach. A priori error estimates are derived. Numerical results validate the theoretical convergence rates.

1. Introduction

The Biot poroelasticity system [1–3] is widely used in several industries such as petroleum and environmental engineering [4,5] and medical applications such as the modeling of the intestinal oedema [6] and cartilage degeneration [7]. The model describes the displacement of a porous medium subject to fluid motion through the pore space.

The differential equations describing the Biot system in the open domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ over the time interval $[0, T]$, are defined as follows

$$\partial_t(c_0 p + \nabla \cdot (\alpha \mathbf{u})) - \nabla \cdot (\kappa \nabla p) = f, \quad \text{in } \Omega \times [0, T], \tag{1}$$

$$-\nabla \cdot (\mathcal{G}e(\mathbf{u}) - \alpha p \mathbf{I}) = \mathbf{g}, \quad \text{in } \Omega \times [0, T]. \tag{2}$$

The displacement, $\mathbf{u}$, of the porous medium and the pressure, p, of the fluid are the primary variables of the Biot system. The problem coefficients are the constrained specific storage coefficient, c_0, and the Biot–Willis coefficient α, with $c_0 \geq 0$ and α in $]0, 1]$. The coefficient κ is the ratio of the permeability coefficient and the fluid viscosity. The system is completed by the following boundary and initial conditions:

$$\mathbf{u} = 0, \ p = 0, \ \text{on } \partial\Omega \times [0, T], \tag{3}$$

$$p = p_0, \ \mathbf{u} = \mathbf{u}_0, \ \text{in } \Omega \times \{0\}. \tag{4}$$

In the momentum equation (2), the stress tensor $\mathcal{G}\mathbf{e}(\mathbf{u})$ is the classical linear elasticity tensor, with Lamé parameters λ, μ:

$$\mathcal{G}\mathbf{e}(\mathbf{u}) = 2\mu \mathbf{e}(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\boldsymbol{I},$$

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and the strain tensor $\mathbf{e}(\mathbf{u})$ is defined as follows

$$\mathbf{e}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T).$$

To solve the Biot system, four approaches have been proposed: *(i)* the implicit approach where the fully coupled system is solved, *(ii)* the loosely explicit where the mechanical response is only updated every few time steps, *(iii)* the iterative scheme where a set of equations is solved iteratively at every time step until a certain tolerance is reached, and *(iv)* the sequential method where the system is completely decoupled and no iterations are needed. The advantage of the latter method is the reduction of the computational cost compared to the fully implicit approach since the method exhibits smaller linear systems to which we can apply standard linear solvers whereas the implicit method might suffer ill-conditioning [8] and special solvers have to be used [9,10]. Furthermore, unlike the iterative scheme, the sequential method does not require several iterations at each time step for convergence. At the continuous level, the convergence of the iterative coupling was proved in [11] and numerical results shown in [12]. Error analysis of the fully implicit approach for various spatial discretizations was carried out in several papers [13–18].

In [19], we analyzed the sequential approach based on a discontinuous Galerkin discretization. We propose to use the sequential approach with the continuous finite element method to discretize the Biot poroelasticity system. We show that by adding stabilization terms as in [19], Eqs. (1) and (2) can be decoupled; and we derive error estimates and therefore extend the sequential approach to the continuous finite element method. This paper follows closely our previous work [19] and the error analysis is based on a similar argument. In [19] we use discontinuous approximations and impose the boundary conditions weakly. In the current paper, the boundary conditions are imposed strongly. Stronger results are obtained in the sense that the stabilization parameter is computable and shown to depend on problem parameters. The choice of the continuous finite element method is justified by its lower computational cost and its simplicity in implementing piecewise linear elements compared to the various face computations of the discontinuous Galerkin method. We also note that the finite element method is widely used to solve the poroelasticity equations [5,20–22] and our proposed formulation can be easily implemented in legacy codes.

An outline of the paper is as follows. We describe the numerical scheme in Section 2. We introduce the projection operators along with preliminary results in Section 3. The error analysis is carried out in Section 4 and is validated in Section 5 numerically.

2. Numerical scheme

We begin by defining the triangulation of Ω and then introduce the numerical scheme. We partition the domain Ω into triangles in $\mathbb{R}^2$ and tetrahedra in $\mathbb{R}^3$ such that h is the maximum diameter over all elements in the mesh, denoted by $\mathcal{E}_h$. We assume that the mesh is nondegenerate *i.e.* there exists a constant $\tau > 0$ independent of h, such that

$$\frac{h_E}{\rho_E} \leq \tau, \quad \forall E \in \mathcal{E}_h,$$

where h_E is the diameter of the element E and ρ_E is the diameter of the largest ball inscribed in E.

Let $k_1 \geq 1$ and $k_2 \geq 1$ be two integers; we define the following discrete spaces

$$\boldsymbol{X}^h = \{\mathbf{v} \in C^0(\Omega)^d\,;\ \forall E \in \mathcal{E}_h,\ \mathbf{v}|_E \in \mathbb{P}_{k_1}(E)^d\},$$
$$M^h = \{q \in C^0(\Omega)\,;\ \forall E \in \mathcal{E}_h,\ q|_E \in \mathbb{P}_{k_2}(E)\},$$
$$\boldsymbol{X}_0^h = \{\mathbf{v} \in \boldsymbol{X}^h\,;\ \mathbf{v} = 0 \text{ on } \partial\Omega\},$$
$$M_0^h = \{q \in M^h\,;\ q = 0 \text{ on } \partial\Omega\},$$

where $\mathbb{P}_k(E)$ is the space spanned by polynomials of degree at most k on the element E.

For the discrete spaces $\boldsymbol{X}_0^h$ and M_0^h, we associate the following energy norms:

$$\|\mathbf{v}\|_X = \|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega)},\ \forall \mathbf{v} \in \boldsymbol{X}_0^h, \tag{5}$$

$$\|q\|_M = \|\kappa^{1/2}\nabla q\|_{L^2(\Omega)},\ \forall q \in M_0^h. \tag{6}$$

The classical Sobolev spaces are denoted by $H^k(\Omega)$ and are equipped with the usual Sobolev norms $\|\cdot\|_{H^k(\Omega)}$ and semi-norms $|\cdot|_{H^k(\Omega)}$. We let $(\cdot,\cdot)_\Omega$ denote the L^2 inner product on the domain Ω and define the following bilinear forms:

$$a(p,q) = (\kappa\nabla p, \nabla q)_\Omega,\ \forall p, q \in H_0^1(\Omega), \tag{7}$$

$$c(\mathbf{u},\mathbf{v}) = (\mathcal{G}\mathbf{e}(\mathbf{u}), \mathbf{e}(\mathbf{v}))_\Omega,\ \forall \mathbf{u}, \mathbf{v} \in (H_0^1(\Omega))^d, \tag{8}$$

$$b(\mathbf{v},q) = -\alpha(\mathbf{v}, \nabla q)_\Omega,\ \forall \mathbf{v} \in (H_0^1(\Omega))^d,\ \forall q \in H_0^1(\Omega). \tag{9}$$

We solve Eqs. (1)–(2) at the discrete times $t_i = i\Delta t$, $i = 0, 1, \ldots, N$ where $\Delta t > 0$ denotes the time step and $t_N = T$. The decoupling approach consists of two steps:

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