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The method of fundamental solutions applied to boundary value problems on the surface of a sphere

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ABSTRACT

In this work we propose using the method of fundamental solutions (MFS) to solve boundary value problems for the Helmholtz–Beltrami equation on a sphere. We prove density and convergence results that justify the proposed MFS approximation. Several numerical examples are considered to illustrate the good performance of the method.

1. Introduction

The numerical solution of boundary value problems (BVPs) for partial differential equations (PDEs) usually relies on methods that require a mesh structure where not only the discretization nodes are required, but also the connections between them. The geometrical discretization plays an important role in these methods, such as the classical finite difference or finite element methods, commonly used to solve BVPs for PDEs.

To avoid the constraints due to mesh based methods, several meshless methods have been proposed that have both advantages and disadvantages. Here we will focus on the Method of Fundamental Solutions (MFS), which can be viewed as a variant of the Boundary Element Method (BEM), that avoids the mesh on the boundary, and singular integration e.g. [1–4]. On the other hand, the MFS presents some difficulties as it generates ill conditioned matrices, and its exponential convergence can only be obtained for analytic boundary data.

In the case of PDEs defined on surfaces of spheres or other manifolds, that appear in problems of applications related to image processing, biology or fluid dynamics [5–7], the use of a geometrical approximation in mesh based methods, brings a new discretization error, that is due to the curvature of the space of the manifold.

Some recent work in these methods include projections onto an approximation of the manifolds from plane triangles [8], the closest point method [9], the orthogonal gradients method involving radial basis functions [10] and others such as surface parametrization [11] and embedding functions [12].

In this work we propose the use of the MFS to solve BVPs when the PDE is defined on the surface of a manifold. This is mainly restricted to the cases where a fundamental solution of the PDE is available, and here this not only depends on the PDE that we are considering, but also on the manifold itself. Therefore we consider the manifold to be a sphere and the Helmholtz equation defined with the Laplace–Beltrami operator, where a known expression of the fundamental solution is available.

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In a recent work [13] the BEM was extended to solve BVPs on the surface of a sphere for the Helmholtz, or Yukawa–Beltrami equation. Therefore we also consider an adaptation of the MFS for the same type of problems, using layer potentials that are defined in this context (see for instance, [13,14]).

In Section 2 we establish some density results that justify the MFS approximation, and a convergence result. In Section 3 we present the algorithm and finally in Section 4 we show the excellent performance of the MFS with some numerical experiments.

2. Dirichlet BVP on manifolds

Let $\bar{\Omega} \subset \mathbb{R}^{d+1}$ be a compact convex set with analytical boundary $\Gamma = \partial\Omega$, and we will consider its boundary that defines a d-dimensional smooth manifold. In this setting we may consider the manifold internal domain $\omega \subseteq \Gamma$, and its complementary domain

$$\omega_c = \Gamma \setminus \bar{\omega},$$

where $\bar{\omega}$ is the closure of ω with respect to the topology induced in Γ.

We will be focusing on the case where Ω is a 3D ball and ω is a non void open surface on the sphere $\partial\Omega$. In this case the boundary of the surface ω is the contour line $\gamma = \partial\omega$, that is non void whenever $\omega \neq \Gamma$, and we assume this contour to be connected and piecewise regular, say C^2.

Now consider the Dirichlet boundary value problem:

$$\begin{cases} (\Delta_\Gamma + \lambda)u = 0, & \text{in } \omega \subset \Gamma, \\ u = g, & \text{on } \gamma = \partial\omega \end{cases} \tag{1}$$

where $\lambda \in \mathbb{C}$ is some constant and Δ_Γ is the Laplace–Beltrami operator acting on Γ, defined by $\Delta_\Gamma u = \nabla_\Gamma \cdot \nabla_\Gamma u$ with $\nabla_\Gamma u = \nabla u - (\mathbf{n} \cdot \nabla u)\mathbf{n}$ being the tangential gradient, where $\mathbf{n}$ stands for the normal to the sphere. The eigenvalues μ_k of the Laplace–Beltrami operator over Γ would lead to non uniqueness issues when $\omega = \Gamma$. A similar situation occurs when $\omega \subset \Gamma$, as problem (1) may lead to non unique solutions if $\lambda = \lambda_k$ is one of the Dirichlet eigenfrequencies associated with Δ_Γ on the domain ω.

3. Density results (Helmholtz case)

The fundamental solution ϕ of the Helmholtz–Beltrami operator verifies

$$(\Delta_\Gamma + \lambda)\phi(x, y) = \delta_\Gamma(x, y), \quad \forall x, y \in \Gamma.$$

This equality with the Dirac delta is understood in the sense of distributions, with test functions $v \in \mathcal{D}(\Gamma)$ (e.g. [15]):

$$\langle D\phi(x, \cdot), v\rangle_{L^2(\Gamma)} = \langle \delta_\Gamma(x, \cdot), v\rangle_{L^2(\Gamma)} = v(x).$$

In the case of the sphere, the fundamental solution is known to be

$$\phi_\lambda(x, y) = \frac{1}{4\sin(\pi V(\lambda))} P_{V(\lambda)}(-x.y), \ x, y \in S^2, \tag{2}$$

with

$$V(\lambda) = \frac{-1 + \sqrt{1 + 4\lambda}}{2}$$

and where P_α are Legendre functions of the first kind (e.g. [16,17]). Note that this fundamental solution is not defined when $V(\lambda) \in \mathbb{N}$, or $\lambda = n(n+1)$, which are the eigenvalues of the Laplace–Beltrami operator for the whole sphere, and for these exceptional cases a generalized fundamental solution can be considered (e.g. [17]).

For instance, when $\lambda = 0$, the fundamental solution of the Laplace–Beltrami operator reduces to

$$\phi_0(x, y) = \frac{1}{4\pi} \log(2 - 2x.y) \tag{3}$$

which is the same formula that we have for the Euclidean plane, taking into account that $\|x - y\|^2 = 2 - 2x.y$, in the unitary sphere.

These fundamental solutions are therefore radial, depending only on the Euclidean distance $r = \|x - y\|$.

3.1. Contour layers

We introduce the notion of contour layers in a similar way as boundary layers, adapting to the case where the domain ω is itself part of a closed Riemannian manifold. Thus, in $\mathbb{R}^3$ these domains ω are parts of (analytic) surfaces, and their boundary becomes a contour γ. With respect to ω a regular C^1 contour γ has an associated normal vector $\nu(x)$, for each $x \in \gamma$. Note that this normal vector ν belongs to the tangent bundle of Γ and points outwards ω.