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Higher-order convergence with fractional-step method for singularly perturbed 2D parabolic convection–diffusion problems on Shishkin mesh

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ABSTRACT

In this article, we propose a second-order uniformly convergent numerical method for a singularly perturbed 2D parabolic convection–diffusion initial–boundary-value problem. First, we use a fractional-step method to discretize the time derivative of the continuous problem on uniform mesh in the temporal direction, which gives a set of two 1D problems. Then, we use the classical finite difference scheme to discretize those 1D problems on a special mesh, which results almost first-order convergence, *i.e.*, $O(N^{-1+\beta}\ln N + \Delta t)$. To enhance the order of convergence to $O(N^{-2+\beta}\ln^2 N + \Delta t^2)$, we use the Richardson extrapolation technique. In support of the theoretical results, numerical experiments are performed by employing the proposed technique.

1. Introduction

In this paper, we consider the following singularly perturbed 2D parabolic convection–diffusion initial–boundary-value problem (IBVP) posed on the domain $\mathfrak{G} = \mathfrak{D} \times \Omega_t$, where $\mathfrak{D} = I_x \times I_y = (0, 1)^2$ and $\Omega_t = (0, T]$:

$$\begin{cases} u_t + \mathcal{L}_\varepsilon u(x, y, t) = f(x, y, t), & (x, y, t) \in \mathfrak{G}, \\ u(x, y, 0) = \phi(x, y), & (x, y) \in \mathfrak{D}, \\ u(x, y, t) = 0, & (x, y, t) \in \partial\mathfrak{D} \times \overline{\Omega}_t, \end{cases} \tag{1}$$

where

$$\mathcal{L}_\varepsilon u = -\varepsilon \Delta u + \mathbf{a}(x, y).\nabla u + b(x, y)u,$$

$0 < \varepsilon \ll 1$ is the perturbation parameter. The coefficients $\mathbf{a} = (a_1, a_2)$ and b are assumed to be sufficiently smooth and bounded functions such that $a_1(x, y) \geq \alpha_x > 0$, $a_2(x, y) \geq \alpha_y > 0$ and $b(x, y) \geq 0$, on $\overline{\mathfrak{D}}$.

Under the sufficient smoothness and necessary compatibility conditions [1] imposed on the functions f and ϕ, the parabolic IBVP (1) admits a unique solution $u(x, y, t)$, which exhibits a regular boundary layer of width $O(\varepsilon)$ along the sides $x = 1$ and $y = 1$, and a corner layer at $(x, y) = (1, 1)$ [2].

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Nomenclature

u	Solution of the model problem (1)
$v,\ w$	Smooth and singular components of u
$\widehat{u}^{n+1}$	Solution of the semidiscrete problem (5)–(6) on Ω_t^M
$\widehat{z}^{n+1}$	Solution of the semidiscrete problem (5)–(6) on Ω_t^{2M}
$\widehat{u}_{extp^t}$	Solution of the semidiscrete problem (5)–(6) on Ω_t^M after time extrapolation
$\widehat{v}^{n+1/2},\ \widehat{w}^{n+1/2}$	Smooth and singular components of $\widehat{u}^{n+1/2}$ on Ω_t^M
$\widehat{\psi}_v^{n+1/2},\ \widehat{\psi}_w^{n+1/2}$	Smooth and singular components of $\widehat{z}^{n+1/2}$ on Ω_t^{2M}
$\widehat{U}^{n+1}$	Solution of (16)–(17) on $\mathfrak{D}^N$ with M mesh intervals in the temporal direction
$\widetilde{U}^{n+1}$	Solution of (16)–(17) on $\mathfrak{D}^{2N}$ with $2M$ mesh intervals in the temporal direction
U	Solution of the fully discrete problem (20)–(21) on $\mathfrak{G}^{N,M}$
$\widehat{V}^{n+1/2},\ \widehat{W}^{n+1/2}$	Smooth and singular components of $\widehat{U}^{n+1/2}$ on $\mathfrak{D}^N$ with M mesh intervals in the temporal direction
$\widehat{\mathcal{V}}^{n+1/2},\ \widehat{\mathcal{W}}^{n+1/2}$	Smooth and singular components of $\widehat{U}^{n+1/2}$ on $\mathfrak{D}^N$ with $2M$ mesh intervals in the temporal direction
$\widehat{V}_{extp^t},\ \widehat{W}_{extp^t}$	Smooth and singular components after time extrapolation on $\mathfrak{D}^N$ with M mesh intervals in the temporal direction
$\widehat{\mathcal{V}}_{extp^t},\ \widehat{\mathcal{W}}_{extp^t}$	Smooth and singular components after time extrapolation on $\mathfrak{D}^{2N}$ with M mesh intervals in the temporal direction
$\widehat{U}_{extp}$	Solution of (16)–(17) after both space and time extrapolation on $\mathfrak{D}^N$ with M mesh intervals in the temporal direction
$\widehat{V}_{extp},\ \widehat{W}_{extp}$	Smooth and singular components of $\widehat{U}_{extp}$ after both space and time extrapolation on $\mathfrak{D}^N$ with M mesh intervals in the temporal direction
U_{extp}	Final extrapolated solution of the fully discrete problem (20)–(21) on $\mathfrak{G}^{N,M}$

This type of problem occurs in several branches of engineering and applied mathematics, such as skin layers in electrical application, edge layers in solid mechanics and boundary layers in fluid mechanics [3]. For example, consider the unsteady incompressible viscous fluid flow problem governed by the Navier–Stokes equation:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \dfrac{1}{Re}\nabla^2\mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0, \end{cases}$$

where $\mathbf{u}$ is the velocity field whose components are u_1, u_2 along x and y directions, and p is the pressure. For sufficiently large value of the *Reynolds number* (Re), the above equation behaves like a singular perturbation problem (SPP). One can observe that the solution of SPP varies rapidly inside the boundary layer region and behaves smoothly in the outer region. Due to such behavior, classical finite difference methods fail to provide satisfactory numerical result on uniform meshes and as a remedy one needs to reduce the spatial step size with respect to ε, to obtain a stable solution, which is computationally expensive.

Two of the most reliable numerical methods for solving such type of problems which are available in the literatures are fitted operator methods (FOMs) and fitted mesh methods (FMMs). In FOMs, one uses an exponentially fitted scheme, which has coefficients of exponential type adapted to the SPP. The extension of FOMs to higher-dimensional problems are too difficult and in some cases it may not be even possible. Whereas, in FMMs, one can use the classical finite difference schemes on the piecewise-uniform (Shishkin) mesh or any other layer-adapted nonuniform meshes. In FMMs, the meshes are fine in the boundary layer regions and coarse in the outer region. More information about the layer-adapted nonuniform meshes and the numerical schemes for SPPs can be found in the books of Farrell et al. [4], Miller et al. [5], and Roos et al. [2].

Due to the presence of boundary layers in SPPs, finding higher-order ε-uniformly convergent numerical solution to SPPs is indeed a difficult task. To obtain even second-order uniform convergence for the case of convection–diffusion SPPs, one has to devise the numerical scheme very cautiously. One cannot simply approximate the convection term by central difference quotient, because it leads to spurious nonphysical oscillations in the numerical solution. Therefore, one has to give some special attention to obtain second-order convergent numerical solutions for SPPs having convection term. One can surpass such difficulty by using the hybrid scheme of [6], or by the Richardson extrapolation technique [7–9].

Since, the problem considered in this article is two-dimensional in space, it can model a physical phenomenon more appropriately. One can find the numerical treatment of such a problem in [10–13], where the authors of [10,11] considered the stationary case and [12,13] are dedicated for time dependent problem. The authors of [14] solved such problem by using the backward-Euler scheme for time derivative and the upwind finite difference scheme for spatial derivatives. But the drawback in this scheme is that, to obtain the numerical solution, one has to handle a banded pentadiagonal matrix at each time step, which is not very efficient in computational perspective. One can overcome such difficulty by using the fraction-step method, which converts the 2D problem into two 1D problems. Clavero et al. [13] used this method to solve