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An optimal compact sixth-order finite difference scheme for the Helmholtz equation☆

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ABSTRACT

In this paper, we present an optimal compact finite difference scheme for solving the 2D Helmholtz equation. A convergence analysis is given to show that the scheme is sixth-order in accuracy. Based on minimizing the numerical dispersion, a refined optimization rule for choosing the scheme's weight parameters is proposed. Numerical results are presented to demonstrate the efficiency and accuracy of the compact finite difference scheme with refined parameters.

1. Introduction

In this paper, we consider the 2D Helmholtz equation

$$\mathcal{L}u := \Delta u + k^2 u = g \tag{1.1}$$

with the wavenumber k, where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the 2D Laplacian, unknown u usually represents a pressure field in the frequency domain, and g denotes the source function. The Helmholtz equation has important applications in many fields of science and engineering, for instance, in aeronautics, geophysics and optical problems. Therefore, it has always been an active field of research to solve the Helmholtz equation more efficiently and accurately (see [1–6] and the reference therein).

For numerically solving the Helmholtz equation, there are mainly finite difference methods (cf. [3,7,8]) and finite element methods (cf. [1,2,9]). As is known to all, the solution of the Helmholtz equation oscillates severely for large wavenumbers, and the quality of the numerical results usually deteriorates as the wavenumber k increasing, which is the so-called "pollution effect" of high wavenumbers [2,9]. For the 2D and 3D Helmholtz equations, the pollution effect of high wavenumbers cannot be eliminated [2]. Due to the pollution effect of high wavenumbers, the wavenumber of the numerical solution is different from the one of the exact solution, which is known as "numerical dispersion" [9]. Due to the numerical dispersion, the numerical method usually requires a finer mesh to ensure the accuracy with the increasing wavenumber. Hence, to discretize the Helmholtz equation, we should pay attention to two issues: one is the numerical dispersion, while the other is the solver cost. In the past years, there is a growing interest in discretization methods where the computational complexity increases only moderately with increasing wavenumber (cf. [1,3,5,10] and the reference therein).

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The finite difference method is a popular and powerful computational method for numerical seismic wave propagation modeling (cf. [3,5,7,8]). It is both easy to implement and computationally efficient. Furthermore, we can improve the accuracy of the solution of the Helmholtz equation by choosing optimal parameters in finite difference formulas [3,5,11]. In recent years, various second-order schemes have been developed for solving the Helmholtz equation [3,5,12]. A rotated 9-point finite difference scheme was introduced for the Helmholtz equation in [3], which is an optimal compact scheme of second-order indeed. In this paper, 9 points were used to approximate the term of zero order in the Helmholtz equation, which is helpful for suppressing the numerical dispersion and improving the accuracy. Recently, the paper [12] presented a dispersion minimizing scheme for the Helmholtz equation based on point-weighting. The methods proposed in [3,5,12] belong to the group of continuous grid finite difference algorithms for wave modeling. For discontinuous grid finite difference algorithms, we refer the interested readers to [11], which presented subgridding finite difference schemes of second-order for the Helmholtz equation with PML. To improve the numerical accuracy, high-order accurate compact finite difference schemes were also considered. The stencil of a compact scheme only has 3×3 points in two dimensions, which is propitious to inverting the resulting matrix and dealing with the boundaries. In [8,13,14], various compact fourth-order methods for the Helmholtz equation were presented. Compact sixth-order finite difference schemes for the 3D Helmholtz equation with constant coefficients were discussed in [15]. For the 2D and 3D Helmholtz equation with variable coefficients, we refer the interested readers to [6,7] to get high-order compact schemes. High-order schemes may enjoy faster convergence rate, when compared to second-order schemes. However, the numerical dispersion still exists for high-order schemes [2]. In this paper, we will further improve the accuracy of the compact sixth-order scheme by minimizing its numerical dispersion.

Our aim is to develop an optimal compact scheme for the 2D Helmholtz equation, which is sixth-order in accuracy. For this purpose, both the convergence order and numerical dispersion of the scheme will be considered. Applying the approach of equation based differencing used in [6,7,14,15], we shall first construct a sixth-order approximation for the Laplacian term. The existing compact sixth-order finite difference schemes usually use one point to approximate the term of zero order in the Helmholtz equation [6,15]. To further suppress the numerical dispersion, we will use 9 points to formulate a sixth-order approximation for the term of zero order. Combining the sixth-order approximation for the Laplacian term with that for the term of zero order leads to a compact finite difference scheme for the Helmholtz equation. The resulting scheme has weight parameters, which can be chosen properly to improve the accuracy of the solution of the Helmholtz equation. Moreover, we will give a convergence analysis of the scheme and show that it enjoys the accuracy of sixth-order. To choose optimal coefficients for finite difference schemes, we will propose an approach only depending on the dispersion equation of the resulting scheme. This method has the advantage of easy implementation.

This paper is organized as follows. In Section 2, we propose a compact finite difference scheme for the 2D Helmholtz equation with constant wavenumbers, and then provide a convergence analysis to show that the scheme is sixth-order in accuracy. To choose optimal weight parameters of the scheme, a refined choice strategy is also proposed. In Section 3, a compact finite difference scheme of sixth-order is presented for the 2D Helmholtz equation with variable wavenumbers. Numerical experiments are given to demonstrate the efficiency and accuracy of the scheme in Section 4. We show that the proposed scheme not only improves the accuracy but also reduces the numerical dispersion significantly. Finally, Section 5 contains the conclusions of this paper.

2. An optimal compact sixth-order finite difference scheme for the Helmholtz equation with constant wavenumbers

In this section, we propose a compact finite difference scheme for the 2D Helmholtz equation with constant wavenumbers. A convergence analysis is then provided to show that the scheme is sixth-order in accuracy. We also present a new refined optimization rule for choosing the scheme's weight parameters based on minimizing the numerical dispersion.

2.1. A compact sixth-order finite difference scheme

We next present a compact finite difference method for the Helmholtz equation, and then prove that the proposed scheme is sixth-order in accuracy.

We turn to constructing a compact sixth-order finite difference scheme for the Helmholtz equation. To describe the finite difference scheme, we consider the network of grid points (x_m, y_n), where $x_m := x_0 + (m-1)h$ and $y_n := y_0 + (n-1)h$. Note that the same step size $h := \Delta x = \Delta y$ is used for both variables x and y. For each m and n, we set $u_{m,n} := u|_{x=x_m, y=y_n}$ and $k_{m,n} := k|_{x=x_m, y=y_n}$. Let $D_x u$ and $D_{xx} u$ (and similarly in the y direction) denote the second-order centered-difference approximations for u_x and u_{xx}, respectively. We also define:

$$
\begin{aligned}
D_x D_y u_{m,n} &:= \frac{1}{4h^2}\left(u_{m+1,n+1} + u_{m-1,n-1} - u_{m-1,n+1} - u_{m+1,n-1}\right),\\
D_y D_{xx} u_{m,n} &:= \frac{1}{2h^3}\left[u_{m+1,n+1} + u_{m-1,n+1} - u_{m+1,n-1} - u_{m-1,n-1} + 2\left(u_{m,n-1} - u_{m,n+1}\right)\right],\\
D_{xx} D_{yy} u_{m,n} &:= \frac{1}{h^4}\{u_{m+1,n+1} + u_{m-1,n-1} + u_{m-1,n+1} + u_{m+1,n-1} + 4u_{m,n}\\
&\quad - 2\left(u_{m,n+1} + u_{m,n-1} + u_{m+1,n} + u_{m-1,n}\right)\},
\end{aligned}
$$