ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The element-free Galerkin method for the nonlinear p-Laplacian equation

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ARTICLE INFO

Article history:
Received 1 November 2017
Received in revised form 13 December 2017
Accepted 22 December 2017
Available online xxxx

Keywords:
Meshless method
Element-free Galerkin method
P-Laplacian equation
Nonlinear iteration
Convergence

ABSTRACT

The element-free Galerkin (EFG) method is developed in this paper for solving the nonlinear p-Laplacian equation. The moving least squares approximation is used to generate meshless shape functions, the penalty approach is adopted to enforce the Dirichlet boundary condition, the Galerkin weak form is employed to obtain the system of discrete equations, and two iterative procedures are developed to deal with the strong nonlinearity. Then, the computational formulas of the EFG method for the p-Laplacian equation are established. Numerical results are finally given to verify the convergence and high computational precision of the method.

1. Introduction

In this paper, we are concerned with the element-free Galerkin (EFG) method [1] for the numerical solution of the following nonlinear p-Laplacian equation [2–10]

$$-\operatorname{div}\left(|\nabla u(\mathbf{x})|^{p-2}\nabla u(\mathbf{x})\right) = f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2)^{\mathrm{T}} \in \Omega, \tag{1}$$

with the Dirichlet boundary condition

$$u(\mathbf{x}) = \bar{u}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_D, \tag{2}$$

and the Neumann boundary condition

$$\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} = \bar{q}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_N, \tag{3}$$

where $p \in (1, \infty)$ is a given parameter, div is the divergence operator, ∇ is the gradient operator, Ω is a two-dimensional bounded domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\mathbf{n} = (n_1, n_2)^{\mathrm{T}}$ is the unit outward normal to Γ, u is the unknown function, f, $\bar{u}$ and $\bar{q}$ are given functions, and

$$|\nabla u(\mathbf{x})| = \sqrt{\nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x})} = \sqrt{\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2}.$$

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https://doi.org/10.1016/j.camwa.2017.12.019

When $p = 2$, Eq. (1) degenerates to the linear Poisson equation. In this case, Eqs. (1)-(3) can be solved by the EFG method easily [11–13]. In this paper, we assume that $p \neq 2$.

The p-Laplacian equation is also known as p-harmonic equation, and is viewed as one of the typical examples of many degenerate nonlinear equations. This equation comes up in the modeling of many nonlinear physical processes such as nonlinear diffusion and filtration [14], power-law materials [15], nonlinear elasticity [16], non-Newtonian flows [17], turbulent flows [18], elastic–plastic torsional creep [19], flows in porous media [20], and image segmentation [21]. Because a very strong nonlinear term is included, it is very difficult to obtain the analytical solutions. Thus, it is important and useful to study the numerical solutions of the p-Laplacian equation. On the other hand, owing to the strong nonlinear characteristic, the numerical approximation of the nonlinear p-Laplacian equation is obviously more arduous than that of the linear Poisson equation.

In the past three decades, considerable attention has been paid to the numerical solutions of the p-Laplacian equation with the finite element method (FEM) [2–7]. In fact, it is believed [4,5] that the p-Laplacian equation contains most of the essential difficulties in studies of the FEM for this class of degenerate nonlinear equations. Besides, the finite difference method (FDM) [8], the finite volume method (FVM) [9] and the discontinuous Galerkin (DG) method [10] have also been used for the numerical solutions of the p-Laplacian equation. However, the performance of the mesh-based methods relies severely on the quality of meshes, which may be time consuming and arduous.

Meshless (or meshfree) methods can get rid of the meshing-related drawbacks by using a set of scattered nodes. Recently, some meshless methods, such as the meshless local Petrov–Galerkin method [22], the finite point method [23], the radial basis functions (RBF) method [24] and the WEB-Spline-based mesh-free finite element method (MFEM) [25] have been used for solving the p-Laplacian equation. Many RBF meshless collocation methods have been proposed to solve partial differential equations [26–28]. The EFG method is a typical meshless method within the framework of Galerkin weak form [1]. In this method, the problem domain is only discretized by scattered nodes to construct approximate solutions with the moving least squares (MLS) approximation [11]. The EFG method has been successfully implemented for various science and engineering problems and thus becomes one of the most important meshless methods [29,30]. However, the application of the EFG method to the p-Laplacian equation has not been found until now.

In this paper, we extend the frontiers of the EFG method into solving the p-Laplacian equation. In the FEM [2–7], the FDM [8], the FVM [9], the DG method [10], the RBF [24] and the MFEM [25], the p-Laplacian equation is reformulated as equivalent minimization problems and then solved with appropriate nonlinear minimization algorithms. Different from these techniques, here the p-Laplacian equation is discretized directly into nonlinear algebraic system by the EFG method. Since the MLS shape function lacks the delta function property, the penalty approach is adopted to enforce the Dirichlet boundary condition. Besides, to deal with the nonlinearity, two iterative procedures are developed. Numerical results indicate that the present EFG method has much less computational error than the FEM and the FVM. Besides, to further illustrate the capability of the current EFG method, numerical results involving a torsional creep problem are given.

The rest of this paper is organized as follows. In Section 2, detailed EFG formulations are deduced for the p-Laplacian equation. Then, numerical examples are presented in Section 3. Finally, some conclusions are contained in Section 4.

2. EFG computational formulas

2.1. Variational formula

In the EFG method, the MLS approximation is used to generate the meshless approximate space. However, the lack of the Kronecker delta function property of the MLS shape functions complicates the enforcement of Dirichlet boundary conditions. Some approaches have been developed to tackle this issue [29,30]. Here, we use the penalty approach. Then, the boundary value problem given by Eqs. (1)-(3) can be approximated as

$$\begin{cases} -\operatorname{div}\left(|\nabla u_\alpha(\mathbf{x})|^{p-2}\nabla u_\alpha(\mathbf{x})\right) = f(\mathbf{x}), & \mathbf{x}\in\Omega, \\ \dfrac{\partial u_\alpha(\mathbf{x})}{\partial \mathbf{n}} + \alpha u_\alpha(\mathbf{x}) = \alpha \bar{u}(\mathbf{x}), & \mathbf{x}\in\Gamma_D, \\ \dfrac{\partial u_\alpha(\mathbf{x})}{\partial \mathbf{n}} = \bar{q}(\mathbf{x}), & \mathbf{x}\in\Gamma_N, \end{cases} \tag{4}$$

where α is a penalty factor. In the case of vanishing Dirichlet condition, i.e., $\Gamma_D = \emptyset$, we have $u_\alpha \equiv u$. Otherwise, the EFG error can be comparable in magnitude to the error due to the enforcement of Dirichlet boundary condition, and a too small or too big penalty factor increases the error. In Refs. [11,12,31,32], satisfactory results have been obtained by choosing $\alpha = 100h^{-2}$, where h denotes the nodal spacing.

According to the first equation in problem (4), we have [2,3]

$$-\int_\Omega \operatorname{div}\left(|\nabla u_\alpha|^{p-2}\nabla u_\alpha\right) v \,\mathrm{d}\Omega = \int_\Omega f v \,\mathrm{d}\Omega, \qquad \forall\, v\in W^{1,p}(\Omega),$$

where $W^{1,p}(\Omega)$ is the usual Sobolev space defined on Ω [33]. Then, applying Gauss formula leads to

$$\int_\Omega |\nabla u_\alpha|^{p-2}\nabla u_\alpha\cdot\nabla v \,\mathrm{d}\Omega - \int_\Gamma |\nabla u_\alpha|^{p-2}\frac{\partial u_\alpha}{\partial \mathbf{n}} v \,\mathrm{d}\Gamma = \int_\Omega f v \,\mathrm{d}\Omega. \tag{5}$$

Please cite this article in press as: X. Li, H. Dong, The element-free Galerkin method for the nonlinear p-Laplacian equation, Computers and Mathematics with Applications (2018), https://doi.org/10.1016/j.camwa.2017.12.019.