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A penalty method for history-dependent variational–hemivariational inequalities[☆]

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ABSTRACT

Penalty methods approximate a constrained variational or hemivariational inequality problem through a sequence of unconstrained ones as the penalty parameter approaches zero. The methods are useful in the numerical solution of constrained problems, and they are also useful as a tool in proving solution existence of constrained problems. This paper is devoted to a theoretical analysis of penalty methods for a general class of variational–hemivariational inequalities with history-dependent operators. Unique solvability of penalized problems is shown, as well as the convergence of their solutions to the solution of the original history-dependent variational–hemivariational inequality as the penalty parameter tends to zero. The convergence result proved here generalizes several existing convergence results of penalty methods. Finally, the theoretical results are applied to examples of history-dependent variational–hemivariational inequalities in mathematical models describing the quasistatic contact between a viscoelastic rod and a reactive foundation.

1. Introduction

Inequality problems arise in a variety of nonlinear models in Physics, Mechanics and Engineering Sciences, cf. [1–9] on variational inequalities, and [10–13] on hemivariational inequalities. Studies of variational inequalities involve arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. Studies of hemivariational inequalities are based on properties of the subdifferential in the sense of Clarke, defined for locally Lipschitz functions, which may be nonconvex. Variational–hemivariational inequalities represent a special class of inequalities, in which both convex and nonconvex functions are present.

Recently, inequalities with history-dependent operators were introduced and studied in connection with contact problems for memory dependent materials. Abstract classes of quasivariational inequalities with history-dependent operators were considered in [14,15] where existence, uniqueness and regularity results were proved. In [16] and [17], a penalty method and numerical analysis of the corresponding inequalities were provided, respectively. Hemivariational inequalities with history-dependent operators were studied in [18–20]. The inequalities in these papers were formulated in the particular case of Sobolev spaces over a bounded domain in $\mathbb{R}^d$ and specific operators like the trace operator. A general existence and

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uniqueness result in the study of variational–hemivariational inequalities with history-dependent operators, on an abstract setting of reflexive Banach spaces, was carried out in the recent paper [21].

Our aim in the current paper is to present a penalty method in the study of history-dependent variational–hemivariational inequalities introduced in [21] and to apply it to a new model of contact. Penalty methods for variational inequalities have been studied by many authors, for numerical purposes and for proofs of solution existence. The reader is referred to [2,3,6] on penalty methods for variational inequalities. The main feature of the penalty methods is that constraints in a problem are enforced by penalty through a limiting procedure and the penalized problems are constraint free. The penalized problems have unique solutions which converge to the solution of the original problem, as the penalty parameter tends to zero. Penalty methods were also considered in [16] and [22] in the study of history-dependent variational inequalities and variational–hemivariational inequalities, respectively. The results in this paper generalize and extend the convergence results obtained in these two papers.

The remainder of the paper is structured as follows. In Section 2, we present the history-dependent variational–hemivariational inequalities, introduce the penalized problems, and state our main theoretical result on solution existence and convergence for the penalty method. The main result is proved in Section 3, in several steps. In Section 4 we present two mathematical models which describe the contact of a viscoelastic rod with a foundation. In the first model the contact is with normal compliance and unilateral constraint, whereas in the second one, the unilateral constraint is penalized. For each model we introduce a weak formulation as a history-dependent variational–hemivariational inequality for the displacement field. Then, we state in Theorem 18 the unique weak solvability of the contact problems and the convergence of the weak solution of the penalized problem to the weak solution of the original problem, as the stiffness coefficient of the foundation converges to infinity. The proof of Theorem 18 is given in Section 5, through an application of the abstract result provided by Theorems 2 and 5 of Section 2.

2. A history-dependent variational–hemivariational inequality

We use real spaces only. For a normed space X, its norm is denoted by $\|\cdot\|_X$. When no confusion may arise, the duality pairing between the dual space X^* and X, $\langle\cdot,\cdot\rangle_{X^*\times X}$, will be simply written as $\langle\cdot,\cdot\rangle$.

We review some definitions. Let $A : X \to X^*$. Then A is monotone if

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq 0 \quad \forall\, v_1, v_2 \in X.$$

It is strongly monotone with a constant $m_A > 0$ if

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A\|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X. \tag{2.1}$$

The operator A is demicontinuous if

$$u_n \to u \text{ as } n \to \infty \quad \Longrightarrow \quad Au_n \to Au \text{ weakly as } n \to \infty.$$

It is hemicontinuous if the function $t \mapsto \langle A(u + t\,v), w\rangle$ is continuous on $[0, 1]$ for all $u, v, w \in X$. Finally, the operator A is pseudomonotone if it is bounded and $u_n \to u$ weakly in X together with $\limsup \langle Au_n, u_n - u\rangle_{X^*\times X} \leq 0$ imply

$$\langle Au, u - v\rangle_{X^*\times X} \leq \liminf \langle Au_n, u_n - v\rangle_{X^*\times X} \quad \forall\, v \in X.$$

A function $\varphi : X \to \mathbb{R}$ is said to be lower semicontinuous (l.s.c.) if for any sequence $\{x_n\} \subset X$, $x_n \to x$ in X implies $\varphi(x) \leq \liminf \varphi(x_n)$.

Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function. Its generalized (Clarke) directional derivative at $x \in X$ in the direction $v \in X$ is defined by

$$\varphi^0(x; v) = \limsup_{y\to x,\ \lambda\downarrow 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}.$$

Its generalized gradient (subdifferential) at x is a subset of the dual space X^*:

$$\partial\varphi(x) = \left\{\zeta \in X^* \mid \varphi^0(x; v) \geq \langle\zeta, v\rangle_{X^*\times X}\ \forall\, v \in X\right\}.$$

It is said to be regular in the sense of Clarke at $x \in X$ if for all $v \in X$, the one-sided directional derivative $\varphi'(x; v)$ exists and $\varphi^0(x; v) = \varphi'(x; v)$.

We use the symbol $\mathbb{N}$ for the set of positive integers and $\mathbb{R}_+ = [0, +\infty)$ for the set of nonnegative real numbers. The notation $C(\mathbb{R}_+; X)$ stands for the space of X-valued continuous functions defined on $\mathbb{R}_+$. For a subset $K \subset X$, $C(\mathbb{R}_+; K) \subset C(\mathbb{R}_+; X)$ denotes the set of K-valued continuous functions defined on $\mathbb{R}_+$.

Given a reflexive Banach space X, $K \subset X$, a normed space Y, operators $A : X \to X^*$ and $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$, a function $\varphi : Y \times X \times X \to \mathbb{R}$, a locally Lipschitz function $j : X \to \mathbb{R}$, and $f : \mathbb{R}_+ \to X^*$, we consider the following problem.

Problem 1. Find $u \in C(\mathbb{R}_+; K)$ such that for all $t \in \mathbb{R}_+$,

$$\begin{aligned}&\langle Au(t), v - u(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), u(t))\\ &\quad + j^0(u(t); v - u(t)) \geq \langle f(t), v - u(t)\rangle \quad \forall\, v \in K.\end{aligned} \tag{2.2}$$