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On high-order conservative finite element methods

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ABSTRACT

We describe and analyze a volumetric and residual-based Lagrange multipliers saddle point reformulation of the standard high-order finite method, to impose conservation of mass constraints for simulating the pressure equation on two dimensional convex polygons, with sufficiently smooth solution and mobility phase. We establish high-order a priori error estimates with locally conservative fluxes and numerical results are presented that confirm the theoretical results.

1. Problem

Many porous media related practical problems lead to the numerical approximation of the pressure equation

$$-\operatorname{div}(\Lambda(x)\nabla p) = q \quad \text{in } \Omega \subset \Re^2, \tag{1}$$

$$p = 0 \quad \text{on } \partial\Omega_D, \tag{2}$$

$$\nabla p \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \setminus \partial\Omega_D, \tag{3}$$

where $\partial\Omega_D$ is the part of the boundary of the domain Ω (denoted by $\partial\Omega$) where the Dirichlet boundary condition is imposed. In case the measure of $\partial\Omega_D$ (denoted by $|\partial\Omega_D|$) is zero, we assume the compatibility condition $\int_\Omega q\,dx = 0$. On the above equation we have assumed without loss of generality homogeneous boundary conditions since we can always reduce the problem to that case. The domain Ω is assumed to be a convex polygonal region in order at least H^2 regularity, see [1], and for a rectangle domain the problem is H^p regular for any integer p. We note however that this convexity or rectangularity is not required for the discretization, they are required only when regularity theory of partial differential equations (PDEs) is considered for establishing the a priori error estimates.

In multi-phase immiscible incompressible flow, p and Λ are the unknown pressure and the given phase mobility of one of the phases in consideration (water, oil or gas); (see e.g., [2–7]). In general, the forcing term q is due to gravity, phase transitions, sources and sinks, or when we transform a nonhomogeneous boundary condition problem to a homogeneous one. The mobility phase in consideration is defined by $\Lambda(x) = K(x)k_r(S(x))/\mu$, where $K(x)$ is the absolute (intrinsic) permeability of the porous media, k_r is the relative phase permeability and μ the phase viscosity of the fluid. The assumptions required in this numerical analysis article may not in general hold for such large-scale flow models.

The main goal of our work is to obtain conservative solution of the equations above when they are discretized by high order continuous piecewise polynomial spaces. The obtained solution satisfies some given set of linear restrictions (may be

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related to subdomains of interest). Our motivations come from the fact that in some applications it is **imperative** to have some conservative properties represented as conservations of total flux in control volumes. For instance, if $\mathbf{q}^h$ represents the approximation to the flux (in our case $\mathbf{q}^h = -\Lambda\nabla p^h$ where p^h is the approximation of the pressure), it is required that

$$\int_{\partial V} \mathbf{q}^h \cdot \mathbf{n} = \int_V q \quad \text{for each control volume } V. \tag{4}$$

Here V is a control volume that does not cross $\partial\Omega_D$ from a set of controls volumes of interest, and here and after $\mathbf{n}$ is the normal vector pointing out the control volume in consideration. If some appropriate version of the total flux restriction written above holds, the method that produces such an approximation is said to be a conservative discretization.

Several schemes offer conservative discrete solutions. These schemes depend on the formulation to be approximated numerically. Among the conservative discretizations for the second order formulation of the elliptic problem we mention the finite volume (FV) method, some finite difference methods and some discontinuous Galerkin methods. On the other hand, for the first order formulation or the Darcy system we have the mixed finite element methods and some hybridizable discontinuous Galerkin (HDG) methods.

In this paper, we consider methods that discretize the second order formulation (1). Working with the second order formulation makes sense especially for cases where some form of high regularity holds. Usually in these cases the equality in the second order formulation is an equality in L^2 so that, in principle, there will be no need to weaken the equality by introducing less regular spaces for the pressure as it is done in mixed formulation with L^2 pressure.

For second order elliptic problems, a very popular conservative discretization is the finite volume (FV) method. The classical FV discretization provides an approximation of the solution in the space of piecewise linear functions with respect to a triangulation while satisfying conservation of mass on elements of a dual triangulation. When the approximation of the piecewise linear space is not enough for the problem at hand, advance approximation spaces need to be used (e.g., for problems with smooth solutions some high order approximation may be of interest). However, in some cases, this requires a sacrifice of the conservation properties of the FV method. Here in this paper, we design and analyze conservative solution in spaces of high order piecewise polynomials. We follow the methodology in [8], that imposes the total flux restrictions by employing Lagrange multiplier technique. This methodology was developed in order to apply the higher-order methods constructed in [9–12] to two-phase flow problems.

We note that FV methods that use higher degree piecewise polynomials have been introduced in the literature. The fact that the dimension of the approximation spaces is larger than the number of restrictions led the researchers to design some method to select solutions: For instance, in [13–15] to introduce additional control volumes to match the number of restrictions to the number of unknowns. It is also possible to consider a Petrov–Galerkin formulation with additional test functions rather that only piecewise constant functions on the dual grid. Other approaches have been also introduced, see for instance [16] and references therein.

In the construction of new methodologies into a reservoir simulation should have into account the following issues: (1) local mass conservation properties, (2) stable-fast solver and (3) the flexibility of re-use of the novel technique into more complex models (such as to nonlinear time-dependent transport equations for the convection dominated transport equation). For Darcy-like model problems with very high contrasts in heterogeneity, the discretization of Darcy-like models alone may be very hard to solve numerically due to a large condition number of the arising stiffness matrix. Moreover, the situation is even more intricate for modeling non trivial two- [17,18] and three-phase [7,6] transport convection dominated phenomena problems for flow through porous media (see also other relevant works [19,2,5,20]). Thus, to achieve a sufficiently coupling between the volume fractions (or saturation) and the pressure–velocity, the full problem can be treated along with a fractional-step numerical procedure [7,6]; we point out that we are aware about the very delicate issues linked to the discontinuous capillary-pressure (see [3] and the references therein). Indeed, the fluxes (Darcy velocities) are smooth at the vertices of the cell defining the integration volume in the dual triangularization, since these vertices are located at the centers of non-staggered cells, away from the jump discontinuities along the edges. This facilitates the construction of second-order and high-order approximations linked to the hyperbolic–parabolic model problem [7,6]. This gives some of the benefits of staggering between primal and dual mesh triangulation by combining our novel high-order conservative finite element method with finite volume for hyperbolic–parabolic conservation laws modeling fluid flow in porous media applications.

Here in this paper, we consider a Ritz formulation and construct a solution procedure that combines a continuous Galerkin-type formulation that concurrently satisfies mass conservation restrictions. We impose finite volume restrictions by using a scalar Lagrange multiplier for each restriction. This is equivalently to a constraint minimization problem where we minimize the energy functional of the equation restricted to the subspace of functions that satisfy the conservation of mass restrictions. Then, in the Ritz sense, the obtained solution is the best among all functions that satisfy the mass conservation restriction.

Another advantage of our formulation is that the analysis can be carried out with classical tools for analyzing approximations to saddle point problems [21]. We analyze the method using an abstract framework and give an example for the case of second order piecewise polynomials. An important finding of this paper is that we were able to obtain optimal error estimates in the H^1 norm as well as the L^2 norm. Our L^2 error analysis requires additional assumptions, including specially collocated dual meshes and $\Lambda = I$, and is obtained by adding the Lagrange multipliers to the approximation p_h by an Aubin–Nitsche trick [22,23].