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Higher degree immersed finite element spaces constructed according to the actual interface

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ABSTRACT

We discuss the construction of higher degree immersed finite element (IFE) spaces that can be used to solve two dimensional second order elliptic interface problems having general interfaces without requiring the mesh to be aligned with the material interfaces. The optimal approximation capability of the proposed piecewise pth degree IFE spaces are demonstrated by numerical experiments with interpolations. Numerical solutions to interface problems generated from a partially penalized method based on the proposed higher order IFE spaces also suggest optimal convergence in both the L^2 and H^1 norms under mesh refinement.

1. Introduction

In this article we report our recent explorations about constructing higher degree immersed finite element (IFE) spaces for solving elliptic interface problems. To be specific and without loss of generality, we consider a bounded domain $\Omega \subset \mathbb{R}^2$ that is separated into two subsets Ω^+ and Ω^- by a curve Γ, see Fig. 1.1. In this domain, we consider the following typical second order elliptic interface problem

$$\begin{cases} -\nabla \cdot (\beta \nabla u) = f, & \text{on } \ \Omega^- \cup \Omega^+, \\ u_{|\partial\Omega} = g, & \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

with

$$\beta = \begin{cases} \beta^-, & \text{in } \Omega^-, \\ \beta^+, & \text{in } \Omega^+ \end{cases}$$

where $\min(\beta^-, \beta^+) > 0$. The exact solution of this model elliptic problem satisfies the following interface jump conditions

$$[u]_\Gamma = 0, \tag{1.2a}$$

$$[\beta\, \boldsymbol{n} \cdot \nabla u]_\Gamma = 0. \tag{1.2b}$$

We also note that when the right-hand side of the elliptic problem is in C^{p-2}, then

$$[\beta \frac{\partial^l \Delta u}{\partial \boldsymbol{n}^l}]_\Gamma = 0, \ \ l = 0, 1, \ldots, p-2. \tag{1.2c}$$

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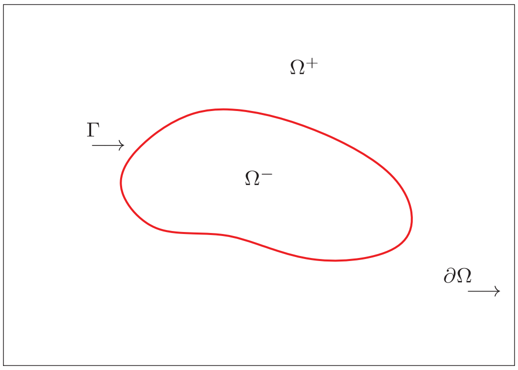

Fig. 1.1. A two-material solution domain Ω.

where the jump $[v]_\Gamma = v^+|_\Gamma - v^-|_\Gamma$ with $v^\pm = v|_{\Omega^\pm}$. The conditions (1.2a) and (1.2b) are called the physical jump conditions while the jump equations (1.2c) are suggested by the smoothness of the right-hand side in (1.1).

Generally speaking, a standard finite element method based on generic problem-independent polynomial shape functions should not have much difficulty, if any, to produce an accurate numerical approximation to the exact solution of the interface problem described by (1.1), (1.2a), and (1.2b) provided that its mesh is "body-fitted" meaning the mesh is formed according to the interface Γ such that each element of this mesh is essentially on one side of Γ. It is well-known [1–3] that, without a "body-fitted" mesh, there is generally no guarantee that the approximate solution produced by a standard finite element method can converge optimally, if it converges at all. This "body-fitted" requirement can hinder applications of standard finite element methods to simulations in which an interface problem has to be repeatedly solved with different interface configurations because it demands a new mesh to be made for each new interface.

On the other hand, immersed finite element (IFE) methods are developed to solve interface problems with interface independent meshes [4–19]. Most elements in the mesh used by an IFE method have no intersection with the interface where generic problem independent polynomial shape functions can still be used as usual. On interface elements, i.e., those elements cut by the interface Γ, an IFE method relies on problem-dependent shape functions constructed according to the jump conditions specified by the interface problem. The problem-dependent shape functions are macro elements, similar to the well-known Hsieh–Clough–Tocher elements [20,21], which are piecewise polynomials patched together by jump conditions. This idea traces back to the generalized finite element method which appeared in 1970s [22,23] and employed shape functions on an element constructed by locally solving the problem in that element, these shape functions may be non-polynomials and were capable of capturing important features of the exact solution.

Since there have been quite a few publications on lower degree (using polynomials of degree 1 or less) IFE methods, developing higher degree IFE spaces is not only the natural next step but also desirable because they can be useful in modern techniques such as the local h and p refinement. Works for one dimensional IFE spaces can be found in [4,6,8,24] while some exploratory work for two dimensional IFE space are in [5,7]. These preliminary works lead us to two essential issues in developing higher degree IFE space. The first issue is a need for extra conditions for determining coefficients of the higher degree polynomials in a higher degree IFE shape function. We note that the physical jump conditions given in an interface problem can be naturally used to uniquely determine a lower degree IFE shape function [10,12,14,15,17], but they are not enough for an IFE shape function constructed with higher degree polynomials. We will show how extended jump conditions such as those given (1.2c) can be employed to augment the given physical jump conditions for constructing higher degree IFE spaces while other types of extended jump conditions are possible [5,7].

The second issue is the interface for IFE functions and how to impose jump conditions on it. For IFE functions constructed by linear or bilinear polynomials, each interface element is partitioned into two sub-elements by a line or a plane approximating the interface Γ and IFE functions on this element are piecewise linear or bilinear polynomial according to these two sub-elements with straight edges. This means the interface of a linear or bilinear IFE function is a polyline across which the jump conditions are naturally applied. While a polyline can approximate a curve interface Γ with $O(h^2)$ accuracy expected from linear or bilinear polynomial, it is not enough to match the desired $O(h^{p+1})$ accuracy when pth degree polynomial is used for IFE functions with $p > 1$. In this article, we propose IFE functions constructed according to the actual interface Γ, not its approximation, and we propose to enforce the interface jump conditions (including those extended ones) by projection to pertinent polynomials spaces through integrals over the interface; hence, the higher degree IFE spaces developed in this article can be applied to more realistic interface problems while those explored in [5,7] can only be employed in special situations such that the interface in each interface element is a strait line.

Essentially, the IFE spaces proposed here are more sophisticated than those in the literature made with lower degree polynomials. It is known, see [10,15] for example, the error estimation even for lower degree IFE spaces demands special complicated techniques because of inapplicability of the powerful scaling argument for standard finite element error

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