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Variational principles for advection–diffusion problems

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ABSTRACT

Variational principles for linear and semilinear advection–diffusion problems with velocity field given by potential flow are described and analyzed. Mixed Dirichlet and prescribed flux conditions are treated. Existence and uniqueness results are proved and equivalent integral operator equations are found. A positive multiplier function related to the potential of the flow is used to change the system to divergence form. The dependence of the solution on inhomogeneous flux boundary data is determined.

1. Introduction

This paper describes variational principles and some results about the solutions of linear and semilinear advection–diffusion equations in bounded regions and subject to mixed Dirichlet and flux boundary conditions. Google searches provide a tremendous number of papers on the numerical simulation of these problems; a paper by Brooks and Hughes [1] has over 5000 citations in Google scholar. These equations arise in some very disparate areas of science and engineering. Most of the papers describe the numerical computation of solutions under different assumptions on the data. There has also been extensive study of algorithms and procedures for efficiently simulating the observed phenomenology. See chapter 5 of Glowinski [2] for a recent discussion of least squares and other algorithms for these systems.

The numerical simulations indicate that, when the advection term is significant, it is better to use methods that respect properties of the velocity field. General existence results for these systems are known based on standard elliptic theory. See section 6.9 of Attouch, Buttazzo and Michaille [3] where existence–uniqueness is proved under an assumption of "slow flow" and zero Dirichlet boundary conditions. Some texts on finite element simulations also prove some existence results.

However there has been relatively little mathematical analysis of problems with physically interesting non-zero boundary conditions that help understand the results observed when the velocity terms are significant. Here the construction and properties of variational principles for the solutions will be treated for velocity fields that are gradient flows.

This boundary value problem is to find solutions of

$$-\,\Delta u + \mathbf{a}\cdot\,\nabla u = f(x,u) \quad \text{on} \quad \Omega \tag{1.1}$$

subject to mixed Dirichlet and flux boundary conditions

$$u = 0 \quad \text{on} \quad \tilde{\Sigma} \quad \text{and} \quad D_\nu u = g(x) \quad \text{on} \quad \Sigma. \tag{1.2}$$

Here $\mathbf{a}:\boldsymbol{\Omega}\to\mathbb{R}^{\mathbf{n}}$ is assumed to be a gradient field on the bounded region $\Omega\subset\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Σ is a proper open subset of the boundary, $\tilde{\Sigma}=\partial\Omega\setminus\overline{\Sigma},\ \nu$ is the unit outward normal at a point on the boundary and f, g are prescribed functions on $\Omega\times\mathbb{R}$, Σ respectively. Further conditions on these functions will be specified below.

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(1.1)–(1.2) are of the form treated by Ortiz [4] who described some quite different variational principles for this problem. Note that if **a** is a non-zero vector field, then (1.1) is not in *divergence form*, nor is it a *potential equation*, that holds at the critical points of a Gateaux-differentiable function on a Sobolev space.

Nevertheless, under the condition that $\mathbf{a} = -\nabla\varphi$ is a gradient field, there is a functional $\mathcal{E}$ on a natural Hilbert Sobolev space $H^1_\Sigma(\Omega)$ whose critical points are precisely the solutions of this system. This functional involves a multiplier (or integrating factor) determined by the potential of the flow field **a**. It appears that this multiplier plays a role analogous to that of preconditioners for the numerical simulation of solutions.

The variational principle is used to prove an existence–uniqueness result for the system subject to simple conditions on the source (or reaction) term and standard assumptions on the region and the boundary conditions. The linear case is described carefully in Section 3 and an equivalent integral formulation is outlined. Section 4 provides representations and approximations of solutions that satisfy the inhomogeneous boundary data. These use a class of mixed Steklov eigenfunctions for the problem.

2. Assumptions and spaces

For the most part, the notation is standard and similar to that used in [3]. The assumptions on the region Ω, its boundary $\partial\Omega$ and the subsets $\Sigma, \tilde{\Sigma}$ are appropriate for the use of finite element models.

(B1) *Ω is a bounded connected open set in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is the union of a finite number of disjoint closed Lipschitz surfaces; each surface having finite surface area and a unit outward normal $\nu(.)$ defined σ a.e.. (When $n = 2$, these are Lipschitz curves of finite length).*

(B2) *Σ is a nonempty open subset of $\partial\Omega$, Σ and $\tilde{\Sigma}$ have strictly positive surface area, $\sigma(\partial\Sigma) = 0$ and $g \in L^{q_T}(\Sigma, d\sigma)$ with $q_T = 2(1 - 1/n)$ for $n \geq 3$ ($q_T > 1$ when $n = 2$).*

The advection term is taken to be a gradient field $\mathbf{a}(x) := -\nabla\varphi(x)$ with *potential* φ. A necessary condition for this is that curl $\mathbf{a} \equiv 0$ on Ω when $n = 2$ or 3. The potential φ should satisfy

(B3) *φ is a Lipschitz ($W^{1,\infty}$) function with $\varphi(x) \geq 0$ on $\overline{\Omega}$.*

Observe that ideal fluid flows are potential flows and the positivity requirement on φ holds without loss of generality since adding a constant to a potential function does not change the gradient. When $\mathbf{a}(\mathbf{x}) \equiv \mathbf{a}$ is a constant field the potential has the form $\varphi(x) = \mathbf{a} \cdot \mathbf{x} + a_0$. When $\mathbf{a}(\mathbf{x}) = \mathbf{2\,c} \cdot \mathbf{x} + \mathbf{c_0}$ is linear, the potential is quadratic; $\varphi(x) = \sum_{j=1}^n c_j x_j^2 + \mathbf{c_0} \cdot \mathbf{x} + a_0$ with a_0 chosen so ensure the positivity of φ on the compact set $\overline{\Omega}$.

Let $\Gamma : H^1(\Omega) \to L^2(\partial\Omega)$ be the boundary trace operator and $H^1_\Sigma(\Omega)$ be the subspace of $H^1(\Omega)$ of all functions that satisfy $\Gamma u(x) = 0$ σ *a.e. on* $\tilde{\Sigma}$. This is a closed subspace of $H^1(\Omega)$ and will be a real Hilbert space with respect to the inner product

$$\langle u, v\rangle := \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \text{since } \Sigma \text{ is non-empty.}$$

A weak formulation of the problem is to find $u \in H^1_\Sigma(\Omega)$ that satisfies

$$a(u, v) := \int_\Omega [\,\nabla u \cdot \nabla v + (\mathbf{a} \cdot \nabla u\,)\, v\,]\, dx = m(u, v) \quad \text{for all} \quad v \in H^1_\Sigma(\Omega). \tag{2.1}$$

Here dx denotes Lebesgue measure and

$$m(u, v) := \int_\Omega f(x, u)\, v\, dx + \int_\Sigma g(x)\, v\, d\sigma \tag{2.2}$$

with $d\sigma$ being surface area measure on the boundary $\partial\Omega$.

Define $\chi(x) := e^{\varphi(x)}$ then $\chi \in W^{1,\infty}(\Omega)$ and $\chi(x) \geq 1$ on $\overline{\Omega}$; χ will be called a *multiplier* for this variational problem.

It is easily verified that u solves (1.1) provided

$$-\,\mathrm{div}(\,\chi\, \nabla u\,) = \chi(x) f(x, u) \quad \text{on} \quad \Omega. \tag{2.3}$$

This equation is in divergence form so there is a variational principle for its solutions. Consider the problem $(\mathcal{P})$ of minimizing $\mathcal{E}$ on $H^1_\Sigma(\Omega)$ where $\mathcal{E} : H^1_\Sigma(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{E}(u) := \int_\Omega \chi \left[\, |\nabla u\,|^2 - 2\, F(x, u)\right] dx - 2\int_\Sigma \chi\, g\, u\, d\sigma \tag{2.4}$$

with $F(x, s) := \int_0^s f(x, t)\, dt$. Note that this functional involves the advection field solely through the multiplier $\chi(x)$. To obtain existence results for this variational problem we require the following condition on the *source* term f and its antiderivative F.

(B4) *Assume $f(., s)$ is Borel measurable on Ω for each $s \in \mathbb{R}$, $f(x, .)$ is continuous on $\mathbb{R}$ for almost all $x \in \Omega$ and for each $\epsilon > 0$ there is a $C(\epsilon)$ such that $F(x, s) \leq C(\epsilon) + \epsilon\, s^2$ on $\Omega \times \mathbb{R}$.*

This is the condition that f is a Caratheodory function whose indefinite integral is sub-quadratic in s. In particular it holds when f is bounded and continuous on $\Omega \times \mathbb{R}$.