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Nonconforming immersed finite element spaces for elliptic interface problems[☆]

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ABSTRACT

In this paper, we use a unified framework introduced in Chen and Zou (1998) to study two nonconforming immersed finite element (IFE) spaces with integral-value degrees of freedom. The shape functions on interface elements are piecewise polynomials defined on sub-elements separated either by the actual interface or its line approximation. In this unified framework, we use the invertibility of the well known Sherman–Morison systems to prove the existence and uniqueness of IFE shape functions on each interface element in either a rectangular or triangular mesh. Furthermore, we develop a multi-edge expansion for piecewise functions and a group of identities for nonconforming IFE functions which enable us to show the optimal approximation capability of these IFE spaces.

1. Introduction

Consider the classical second-order elliptic interface problem:

$$-\nabla\cdot(\beta\nabla u)=f,\quad \text{in } \Omega^-\cup\Omega^+, \tag{1.1}$$

$$u=g,\quad \text{on } \partial\Omega, \tag{1.2}$$

where the domain $\Omega \subseteq \mathbb{R}^2$ is assumed to be separated by an interface curve Γ into two subdomains Ω^+ and Ω^-. The diffusion coefficient $\beta(X)$ is a piecewise constant:

$$\beta(X)=\begin{cases}\beta^- & \text{if } X\in\Omega^-,\\ \beta^+ & \text{if } X\in\Omega^+,\end{cases}$$

and the exact solution u is required to satisfy the jump conditions:

$$[u]_\Gamma = 0, \tag{1.3}$$

$$\left[\beta\nabla u\cdot\mathbf{n}\right]_\Gamma = 0, \tag{1.4}$$

where $\mathbf{n}$ is the unit normal vector to the interface Γ. Here and from now on, for every piecewise function v defined as

$$v=\begin{cases}v^-(X) & \text{if } X\in\Omega^-,\\ v^+(X) & \text{if } X\in\Omega^+,\end{cases}$$

we adopt the notation $[v]|_\Gamma = v^+|_\Gamma - v^-|_\Gamma$.

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The IFE method was introduced in [1] for solving an 1D elliptic interface problem with meshes independent of the interface. Extensions to 2D elliptic interface problems include IFE functions defined by conforming P_1 polynomials [2–6], conforming Q_1 polynomials [7–10], nonconforming P_1 (Crouzeix–Raviart) polynomials [11], and nonconforming rotated-Q_1 (Rannacher–Turek) polynomials [12–14]. IFE shape functions in these articles are H^1 functions defined with a line approximating the original interface curve in each interface element. Recently, the authors in [15,16] developed IFE spaces according to the original interface curve where the local degrees of freedom are of Lagrange type. The goal of this article is to develop and analyze IFE spaces constructed with the actual interface curve and the degrees of freedom as the integral values on element edges.

There are two motivations for us to consider IFE functions with integral-value degrees of freedom and with the actual interface curve instead of its line approximation. First, as observed in [13,14], IFE functions of this non-Lagrange type usually have less severe discontinuity across interface edges because their continuity across an element edge is weakly enforced over the entire edge in an average sense. Compared to Lagrange type IFE spaces, the IFE spaces with integral-value degrees of freedom, such as the one considered in [13], usually do not require additional penalty terms in order to obtain accurate approximation in both the actual computation and the error analysis. This important feature is corroborated by numerical examples in this article and [14].

The second motivation is our desire to develop higher degree IFE spaces for which using a line to approximate the interface curve is not sufficient anymore because of the $O(h^2)$ accuracy limitation for the line to approximate a curve. Recently, we have constructed high order immersed finite element spaces based on curve partition using the least squares method [17]. Even though the analysis for IFE functions in this article is still for lower degree nonconforming P_1 or rotated-Q_1 polynomials, we hope our investigation can serve as a precursor to the development of higher degree IFE spaces. In addition, we will demonstrate later that the framework presented here can also be applied to nonconforming IFE spaces based on the line partitioning [11,13,14].

Even though the new IFE spaces presented here seem to be natural because they are constructed locally on each interface element according to the actual interface curve of the problem to be solved, the related investigation faces a few hurdles. The first one is that the new IFE functions are discontinuous in each interface element except for trivial interface geometry because, in general, two distinct polynomials cannot perfectly match each other on a curve. In contrast, almost all IFE spaces in the literature are continuous in each element. This lack of continuity leads to a lower regularity of IFE functions in interface elements such that related error analysis demands new approaches different from those in the literature [5,8,14,18,19]. Another issue is that the interpolation error analysis technique based on the multi-point Taylor expansion in the literature is not applicable here because new IFE functions are constructed with integral-value degrees of freedom instead of the Lagrange type degrees of freedom.

The rest of this article is organized as follows. In Section 2, we introduce some basic notations, assumptions and known results to be used in this article. In Section 3, we extend the multi-point Taylor expansion established in [5,7,14,16] to a multi-edge expansion for piecewise C^2 functions such that the new expansion can handle integral-value degrees of freedom. Estimates for remainders in this new expansion are also derived in this section. In Section 4, we show that the integral-value degrees of freedom imposed on each edge and the approximated jump conditions together yield a Sherman–Morrison system for determining coefficients in an IFE shape function on interface elements. We show that the unisolvence and boundedness of IFE shape functions follow from the well-known invertibility of the Sherman–Morrison system. A group of fundamental identities such as partition of unity are also derived for new IFE shape functions. In Section 5, we establish the optimal approximation capability for IFE spaces with the integral-value degrees of freedom defined either according to the actual interface or to a line approximating the interface curve [11,14]. In Section 6, we present some numerical examples.

2. Preliminaries

Throughout this article, we adopt the notations used in [16], and we recall some of them for reader's convenience. We assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain that is a union of finitely many rectangles, and that Ω is separated by an interface curve Γ into two subdomains Ω^+ and Ω^- such that $\overline{\Omega} = \overline{\Omega^+} \cup \overline{\Omega^-} \cup \Gamma$. For any measurable subset $\tilde{\Omega} \subseteq \Omega$, we consider the standard Sobolev spaces $W^{k,p}(\tilde{\Omega})$ and the associated norm $\|\cdot\|_{k,p,\tilde{\Omega}}$ and semi-norm $|v|_{k,p,\tilde{\Omega}}$. The corresponding Hilbert space is $H^k(\tilde{\Omega}) = W^{k,2}(\tilde{\Omega})$. When $\tilde{\Omega}^s = \tilde{\Omega} \cap \Omega^s \neq \emptyset$, $s = \pm$, we let

$$PW^{k,p}_{int}(\tilde{\Omega}) = \{u : u|_{\tilde{\Omega}^s} \in W^{k,p}(\tilde{\Omega}^s),\ s = \pm;\ [u] = 0 \text{ and } [\beta \nabla u \cdot \mathbf{n}_\Gamma] = 0 \text{ on } \Gamma \cap \tilde{\Omega}\},$$

$$PC^{k}_{int}(\tilde{\Omega}) = \{u : u|_{\tilde{\Omega}^s} \in C^{k}(\tilde{\Omega}^s),\ s = \pm;\ [u] = 0 \text{ and } [\beta \nabla u \cdot \mathbf{n}_\Gamma] = 0 \text{ on } \Gamma \cap \tilde{\Omega}\},$$

with the associated norms and semi-norms:

$$\|\cdot\|^p_{k,p,\tilde{\Omega}} = \|\cdot\|^p_{k,p,\tilde{\Omega}^+} + \|\cdot\|^p_{k,p,\tilde{\Omega}^-}, \qquad |\cdot|^p_{k,p,\tilde{\Omega}} = |\cdot|^p_{k,p,\tilde{\Omega}^+} + |\cdot|^p_{k,p,\tilde{\Omega}^-}.$$