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A fast finite difference method for distributed-order space-fractional partial differential equations on convex domains

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ABSTRACT

Fractional partial differential equations (PDEs) provide a powerful and flexible tool for modeling challenging phenomena including anomalous diffusion processes and long-range spatial interactions, which cannot be modeled accurately by classical second-order diffusion equations. However, numerical methods for space-fractional PDEs usually generate dense or full stiffness matrices, for which a direct solver requires $O(N^3)$ computations per time step and $O(N^2)$ memory, where N is the number of unknowns. The significant computational work and memory requirement of the numerical methods makes a realistic numerical modeling of three-dimensional space-fractional diffusion equations computationally intractable.

Fast numerical methods were previously developed for space-fractional PDEs on multidimensional rectangular domains, without resorting to lossy compression, but rather, via the exploration of the tensor-product form of the Toeplitz-like decompositions of the stiffness matrices. In this paper we develop a fast finite difference method for distributed-order space-fractional PDEs on a general convex domain in multiple space dimensions. The fast method has an optimal order storage requirement and almost linear computational complexity, without any lossy compression. Numerical experiments show the utility of the method.

1. Introduction

In the last few decades many diffusion processes were found to be non-Fickian, ranging from the design of copiers and laser printers [1] and plasma turbulence [2] to solute transport in groundwater [3,4] and other applications [5,6]. In these processes, a particle's motion may have long-range interactions and the probability density function of finding a particle somewhere in space decays algebraically. These processes are characterized by a Lévy distribution whose probability density distribution function satisfies a fractional diffusion equation [7]. This partially explains why fractional PDEs provide a powerful and flexible tool for modeling challenging phenomena including anomalous diffusion processes and long-range spatial interactions, which cannot be modeled accurately by classical integer-order PDEs.

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However, fractional PDEs present severe numerical difficulties. Because of the non-local nature of fractional differential operators, the numerical methods for space-fractional PDEs usually generate dense or even full stiffness matrices [8–12], for which a direct solver requires $O(N^3)$ computations per time step and $O(N^2)$ memory, where N is the number of unknowns. The significant computational work and memory requirement of the numerical methods makes a realistic numerical modeling of three-dimensional space-fractional diffusion equations computationally intractable.

To date, many fast numerical methods were developed for space-fractional PDEs on multidimensional rectangular domains [13–18], not resorting to lossy compression, but rather, via a careful analysis of the structure of the stiffness matrix of the numerical methods. It was proved that the stiffness matrices of the numerical methods have certain tensor product forms of Toeplitz-like structures. Consequently, the coefficient matrices can be stored in $O(N)$ memory and the matrix–vector multiplication can be carried out in $O(N \log N)$ computations via fast Fourier transform (FFT) without any lossy compression. Consequently, fast Krylov subspace iterative methods can be developed. Numerical experiments show the utility of the fast methods. However, so far these fast methods apply only to space-fractional PDEs on rectangular domains only.

In this paper we develop a fast finite difference method for variable-coefficient space-fractional FPDEs, multi-term space-fractional PDEs, and distributed-order space-fractional PDEs on a general convex domain in multiple space dimensions. The stiffness matrices of these numerical methods do not have a block–Toeplitz–Toeplitz–block like structure any longer. Hence, the idea in [15] does not apply. Instead, we utilize the symmetry of fractional derivatives with respect to x and y in the governing fractional diffusion equations. We then borrow the idea of relabeling in alternating-direction implicit finite difference method [12,17,18] to develop a fast matrix–vector multiplication mechanism via a relabeling of the nodal indices of the numerical solution. This fast algorithm still stores the coefficient matrix in $O(N)$ memory and carry out the matrix–vector multiplication in $O(N \log N)$ computations, where N is the number of interior nodes within the convex domain. In other words, the fast algorithm stores the coefficient matrix and carry out the matrix–vector multiplication as efficiently as the algorithm for a rectangular domain, despite that a different mechanism is employed. Furthermore, we emphasize that the fast matrix–vector multiplication algorithm does not involve any splitting of the numerical scheme or lossy compression as in the context of alternating-direction implicit finite difference scheme. Therefore, the fast finite difference method is not lossy.

The remainder of the paper is organized as follows. In Section 2 we present variable-coefficient space-fractional PDEs, multi-term space-fractional PDEs and distributed-order space-fractional PDEs in a general convex domain in the plane, which are to be solved in this paper. In Section 3 we present the corresponding finite difference schemes for these problems. We also decompose the stiffness matrix A^m as the sum of two matrices $A^{m,x}$ and $A^{m,y}$, which account for the couplings of the finite difference schemes in the x and y directions, respectively. In Section 4 we develop an efficient $O(N)$ storage mechanism for $A^{m,x}$. We also design a fast $O(N \log N)$ algorithm for evaluating the matrix–vector multiplication by $A^{m,x}$. In Section 5 we utilize the symmetry of the space-fractional derivatives in the x and y directions and an index relabeling technique to develop an efficient $O(N)$ storage mechanism for $A^{m,y}$ and a fast $O(N \log N)$ algorithm for evaluating the matrix–vector multiplication by $A^{m,y}$. In Section 6 we present a fast Krylov subspace iterative algorithm for the efficient solution of the finite difference schemes, based upon the efficient storage of A^m and the fast matrix–vector multiplication by A^m. In Section 7 we carry out numerical experiments to investigate the performance of the fast finite difference methods, which show the utility of the fast methods developed in this paper. In Section 8 we draw concluding remarks and outline the extension to three-dimensional problems.

2. Problem formulation

We consider the initial–boundary value problem of the time-dependent distributed-order space-fractional PDE on a two-dimensional convex domain:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \Big[d_+(x,y,t)\,{}_{a_1(y)}\mathbb{D}_x^{p_1(\alpha)}u(x,y,t) + d_-(x,y,t)\,{}_x\mathbb{D}_{b_1(y)}^{p_1(\alpha)}u(x,y,t)\Big] \\
&\quad - \Big[e_+(x,y,t)\,{}_{a_2(x)}\mathbb{D}_y^{p_2(\beta)}u(x,y,t) + e_-(x,y,t)\,{}_y\mathbb{D}_{b_2(x)}^{p_2(\beta)}u(x,y,t)\Big] \\
&\qquad = f(x,y,t), \quad (x,y)\in\Omega, \quad t\times(0,T], \\
&u(x,y,0) = u_0(x,y), \quad (x,y)\in\Omega, \\
&u(x,y,t) = 0, \quad (x,y)\in\partial\Omega, \quad t\in[0,T].
\end{aligned}
\tag{1}
$$

Here Ω is a bounded convex domain in the plane. $a_1(y)$ and $b_1(y)$ represent the left and right boundaries of Ω at given y, and $a_2(x)$ and $b_2(x)$ represent the lower and upper boundaries of Ω at given x. The distributed-order space-fractional differential