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A new functional for variational mesh generation and adaptation based on equidistribution and alignment conditions

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ABSTRACT

A new functional is presented for variational mesh generation and adaptation. It is formulated based on combining the equidistribution and alignment conditions into a single condition with only one dimensionless parameter. The functional is shown to be coercive but not convex. A solution procedure using a discrete moving mesh partial differential equation is employed. It is shown that the element volumes and altitudes of a mesh trajectory of the mesh equation associated with the new functional are bounded away from zero and the mesh trajectory stays nonsingular if it is so initially. Numerical examples demonstrate that the new functional performs comparably as an existing one that is also based on the equidistribution and alignment conditions and known to work well but contains an additional parameter.

1. Introduction

Variational mesh generation and adaptation has proven a useful tool in the numerical solution of partial differential equations (PDEs); e.g., see [1–5] and references therein. In this, a (adaptive) mesh is generated as the image of a reference mesh under a coordinate transformation which is determined as the minimizer of a meshing functional. One of the main advantages of this approach of mesh generation is that different mesh requirements such as smoothness, orthogonality, adaptivity, alignment, etc. can easily be incorporated into the formulation of the meshing functional [6]. In addition to being a method for mesh generation and adaptation, this approach can also be used as a smoothing device for automatic mesh generation [7,8] and a base for adaptive moving mesh methods [2,9–11].

There exists a vast literature on variational mesh generation and adaptation. A number of meshing functionals have been developed from different problems and formulated based on different focused requirements. For example, Winslow [12] develops an equipotential method that is based on variable diffusion. Brackbill and Saltzmann [6] combine mesh concentration, smoothness, and orthogonality to create a functional. Dvinsky [13] develops a method based on the energy of harmonic mappings. Knupp [14] and Knupp and Robidoux [15] focus on the idea of conditioning the Jacobian matrix of the coordinate transformation. Huang [16] and Huang and Russell [2] have proposed two methods based on the so-called equidistribution and alignment conditions.

Compared to the algorithmic development, very few theoretical results are known [17–19]. For example, Dvinsky's meshing functional [13] is guaranteed to have a unique invertible minimizer by the theory of harmonic mappings between multidimensional domains. Winslow's functional [12] is known to have a unique minimizer due to its uniformly convexity

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and coercivity. Furthermore, the functional by Huang [16] is coercive and polyconvex and thus has minimizers [2]. Recently, a new formulation of the so-called moving mesh partial differential equation (MMPDE) method [9,10] was proposed by Huang and Kamenski [20], where the meshing functional is first discretized and then the mesh equation (which will be referred to as the discrete MMPDE hereafter) is defined as a gradient system of the discretized functional. This new formulation provides an explicit, compact, and analytical formula for the mesh velocity, which makes the implementation of the method much easier and more robust (cf. Section 3). More importantly, several important properties of the discrete MMPDE can be established; see [21] and/or Section 4 for detail. In particular, the mesh trajectory of the discrete MMPDE stays nonsingular if it is so initially provided that the meshing functional under consideration satisfies a coercivity condition (cf. (16) below). To our best knowledge, this is the only nonsingularity result at the discrete level available in the context of variational mesh generation and adaptation and mesh movement.

It is noted that the functional of [16] satisfies the coercivity condition for a large range of its parameters. It works well with the framework of MMPDEs and has been successively used for various applications [2]. The functional is formulated based on the equidistribution and alignment conditions—more precisely, based on an averaging of the two conditions with a dimensionless parameter. Although the performance of the functional does not seem sensitive to the value of the parameter, its choice is still arbitrary and there is hardly a convincing guideline for choosing it.

The objective of this paper is to present a new functional using the equidistribution and alignment conditions. Like the existing functional of [16], this new one is also based on a combination of the two conditions into a single one, but this time, without introducing any new parameter. We will show that the new functional satisfies the coercivity condition and has similar theoretical properties as the existing functional when employed with the MMPDE. Two-dimensional numerical results will be presented to verify theoretical findings as well as demonstrate comparable performances of the two functionals.

It is worth pointing out that variational mesh adaptation is a special type of anisotropic mesh adaptation which has become an area of intensive research. There is a vast literature in this area; for example, some of the earlier works are [22–34].

An outline of this paper is as follows. In Section 2, the equidistribution and alignment conditions will be introduced and the existing and new functionals will be described. The discrete MMPDE will be presented as a solution procedure for the minimization problem associated with a meshing functional in Section 3. Section 4 is devoted to the study of the theoretical properties of the new functional, followed by the numerical examples in Section 5. Finally, Section 6 contains conclusions and further comments.

2. Meshing functionals based on equidistribution and alignment conditions

In this section we are going to describe two meshing functionals that are formulated from the equidistribution and alignment conditions (cf. (2) and (3) below). These conditions have been developed based on the concept of uniform meshes in some metric tensor [2]. They provide total control of the mesh element size, shape, and orientation of mesh elements through a metric tensor. One of the meshing functionals to be described was first introduced in [16] and involves averaging functionals associated with the two conditions. It has a number of advantages (which will be discussed later) and is known to work well in practice but involves two dimensionless parameters. The second functional is new. It is formulated by directly combining the equidistribution and alignment conditions into a single condition which in turn has eliminated one of the two parameters of the existing functional.

2.1. The equidistribution and alignment conditions

Let the physical domain, $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded (not necessarily convex) polygonal or polyhedral domain and $\mathbb{M} = \mathbb{M}(\boldsymbol{x})$ be a given symmetric, uniformly positive definite metric tensor defined on Ω which satisfies

$$\underline{m} I \leq \mathbb{M}(\boldsymbol{x}) \leq \overline{m} I, \quad \forall\, x \in \Omega, \tag{1}$$

where $\underline{m}$ and $\overline{m}$ are positive constants and I is the identity matrix. Our goal is to generate a simplicial mesh for Ω which is uniform with respect to the metric $\mathbb{M}$. Denote this target mesh by $\mathcal{T}_h = \{K\}$ and let N and N_v be the number of its elements and vertices, respectively. Assume that the reference element $\hat{K}$ has been chosen to be equilateral and unitary (i.e., $|\hat{K}| = 1$, where $|\hat{K}|$ denotes the volume of $\hat{K}$). For any element $K \in \mathcal{T}_h$ let $F_K : \hat{K} \to K$ be the affine mapping between them and $F_K^{'}$ be its Jacobian matrix. Denote the vertices of K by $\boldsymbol{x}_j^K$, $j = 0, \ldots, d$ and the vertices of $\hat{K}$ by $\boldsymbol{\xi}_j$, $j = 0, \ldots, d$. Then

$$\boldsymbol{x}_j^K = F_K(\boldsymbol{\xi}_j).$$

With this in mind, we can define the equidistribution and alignment conditions that completely characterize a non-uniform mesh. Indeed, any non-uniform mesh can be viewed as a uniform one in some metric tensor. Using this viewpoint it is shown (e.g., see [2]) that a uniform mesh in the metric $\mathbb{M}$ satisfies

$$\text{equidistribution} : |K| \det(\mathbb{M}_K)^{\frac{1}{2}} = \frac{\sigma_h}{N}, \quad \forall\, K \in \mathcal{T}_h \tag{2}$$

$$\text{alignment} : \frac{1}{d} \operatorname{tr}\left((F_K^{'})^{-1} \mathbb{M}_K^{-1} (F_K^{'})^{-T} \right) = \det\left((F_K^{'})^{-1} \mathbb{M}_K^{-1} (F_K^{'})^{-T} \right)^{\frac{1}{d}}, \quad \forall\, K \in \mathcal{T}_h \tag{3}$$