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Well-balanced finite difference weighted essentially non-oscillatory schemes for the Euler equations with static gravitational fields

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ABSTRACT

Euler equations of compressible gas dynamics, coupled with a source term due to the gravitational fields, often appear in many interesting astrophysical and atmospheric applications. In this paper, we design high order finite difference weighted essentially non-oscillatory (WENO) methods for the Euler equations under static gravitation fields, which are well-balanced for known steady state solutions. We simplify the well-balanced WENO methods designed in Xing and Shu (2013) for the isothermal equilibrium, and then extend them to more general steady state solutions which include both isothermal and polytropic equilibria. One- and two-dimensional numerical examples are provided at the end to test the performance of the proposed WENO methods and verify these properties numerically.

1. Introduction

In this paper, we design high order finite difference weighted essentially non-oscillatory (WENO) methods for the Euler equations under static gravitation fields, which are well-balanced for known steady state solutions. The mathematical model under consideration is the Euler equations of compressible gas dynamics, coupled with a source term due to the gravitational fields. This is an important model appearing in many interesting astrophysical and atmospheric applications. It takes the form of

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\\ &(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u} + p\mathbf{I}_d) = -\rho\nabla\phi,\\ &E_t + \nabla\cdot((E+p)\mathbf{u}) = -\rho\mathbf{u}\cdot\nabla\phi, \end{aligned} \tag{1}$$

where $\mathbf{x}\in\mathbb{R}^d$ $(d = 1, 2, 3)$ is the spatial variable, and t is the time. Here $\rho, \mathbf{u}, p$ stand for the fluid density, the velocity, and the pressure, respectively. In this paper, $\phi = \phi(\mathbf{x})$ denotes the time independent gravitational potential, and we leave the case of time dependent ϕ for future research. $\mathbf{I}_d$ is the identity matrix, and the operators ∇, $\nabla\cdot$ and $\otimes$ are the gradient, divergence and tensor product in $\mathbb{R}^d$, respectively. $E = \rho\|\mathbf{u}\|^2/2 + \rho e$ (with e being the internal energy) is the non-gravitational energy. To close the system, the pressure p is linked to ρ and e through the so called equation of state, denoted by $p = p(\rho, e)$. The ideal gas law, given by

$$p = (\gamma-1)\rho e = (\gamma-1)\left(E - \rho\|\mathbf{u}\|^2/2\right), \tag{2}$$

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where γ is the ratio of specific heats, is used in the numerical tests, but we would like to mention that the methods presented in this paper are applicable beyond the ideal gas equation of state.

Many existing numerical methods, including finite difference, finite volume and finite element discontinuous Galerkin (DG) methods, have been designed for the Euler equations without the source term due to gravitational fields. However, new numerical challenge emerges with the addition of the gravitational source term, as such model now admits non-trivial steady state solutions with two well-known hydrostatic steady states being the isothermal and the polytropic equilibria. Many practical applications involve the nearly steady state solutions, which are often small perturbations of these steady states. Standard numerical methods fail to balance the numerical approximations of the flux term and the source term at steady state, and introduce error comparable with the size of the physical interesting perturbation, which leads to non-physical oscillating solutions. Well-balanced methods, which can preserve the steady state solution exactly in the discrete level, are introduced to provide an efficient way to capture these small perturbations on relatively coarse meshes. Well-balanced methods for the Euler equations with gravitation were first considered by LeVeque and Bale [1], where they extended their quasi-steady wave propagation methods designed for the shallow water equations to the Euler equations. After this pioneer work, many well-balanced methods [2–8] have been designed and studied. Among them, we designed high order well-balanced finite difference WENO methods for the isothermal equilibrium in [9]. Other high order well-balanced methods can be found in [10–13], which include finite difference, finite volume and finite element DG methods.

In this paper, we propose to extend the well-balanced finite difference WENO methods designed by us in [9] for the isothermal equilibrium to more general steady state solutions which include both isothermal and polytropic equilibria. The main idea to achieve the well-balanced property is to rewrite the source terms in a special way using the information of the equilibrium state to be preserved and then discretize them using a WENO differential operator which is consistent to that for the flux term. This technique has been applied in [14–16] for the shallow water equations, and later generalized for various models including the pollutant transport equations, chemosensitive movement equation, nozzle flow, Euler equations with gravitation, and chemical reaction problems, etc. The major contribution of this paper is to first simplify the source term evaluation of the well-balanced methods in [9] which will lead to a reduced computational cost, and then extend this technique to the polytropic and other general steady state solution of the Euler equations under gravitation.

This paper is organized as follows. In Section 2, we start with presenting the mathematical model and the corresponding steady state solutions, and then propose a novel one-dimensional high order well-balanced WENO scheme, which can conserve the equilibrium state solutions exactly, and at the same time is genuinely high order accurate for the general solutions. We then extend the proposed well-balanced method to multi-dimensional problems in Section 3. Section 4 contains extensive one- and two-dimensional numerical results to demonstrate the behavior of the proposed methods, such as high order accuracy, the well-balanced property, and good resolution for smooth and discontinuous solutions. Conclusion remarks are given in Section 5.

2. Well-balanced WENO schemes for one-dimensional problems

In this section, we present our high order well-balanced finite difference WENO schemes for the Euler equations with gravitation. The simpler one-dimensional problem is discussed in this section to illustrate the main idea, and their generalization to multi-dimensional case will be given in Section 3.

2.1. Mathematical model and steady states

In one spatial dimension, the Euler equations under gravitation (1) become

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + (\rho u^2 + p)_x = -\rho \phi_x, \\ &E_t + ((E + p)u)_x = -\rho u \phi_x, \end{aligned} \tag{3}$$

where u is the one-dimensional velocity. The hydrostatic equilibrium state with a zero velocity is given by

$$u = 0, \quad p_x = -\rho \phi_x, \tag{4}$$

where the source term due to the gravitational forces is balanced by the pressure gradient. Two important special equilibria, which correspond to constant temperature (isothermal equilibrium) and constant entropy (polytropic equilibrium), are commonly encountered in the practical applications.

The isothermal equilibrium state with the constant temperature T_0 is given by

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT_0}\right), \qquad u = 0, \qquad p = p_0 \exp\left(-\frac{\phi}{RT_0}\right), \tag{5}$$

where $p_0 = \rho_0 RT_0$ and R is the gas constant. The polytropic hydrostatic equilibrium usually takes the form of

$$p = K\rho^{\gamma}. \tag{6}$$

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